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MATHEMATICS

Axiomatic Characterization of some Rings of Real Functions

by

W. ZAWADOWSKI

Presented by A. MOSTOWSKI on March 20, 1958

Let R be a commutative ring and let $\geqslant 0$ be a relation in R satisfying the following conditions:

1α. $a \geqslant 0$ and $b \geqslant 0$ implies $ab \geqslant 0$ and $a+b \geqslant 0$;
2α. if $a \geqslant 0$ and $b \geqslant 0$ and either a or b is thick *), then so is $a+b$;
3α. $a^2 \geqslant 0$ for each $a \in R$;
4α. if b is thick and $b+na \geqslant 0$ for $n = 1, 2, \ldots$, then $a \geqslant 0$;
5α. $a \geqslant 0$ and $-a \geqslant 0$ implies $a = 0$.

We write $a \leqslant b$ if, and only if, $b-a \geqslant 0$. The ring R satisfying this system of axioms which contains at least one thick element is called an α-real ring. We will make an extension of an α-real ring to a ring of another kind, which will be called a β-real ring.

Let S be the set of all thick elements of an α-real ring R. We construct the ring (R, S) as follows: (R, S) is the set of all classes (a/s) with respect to the equivalence relation: $a/s \equiv a'/s'$ if, and only if, $as' = a's$, where $a, a' \in R$ and $s, s' \in S$. The addition and multiplication are defined as usual among fractions: $(a/s)+(b/r) = (ar+bs/sr)$, $(a/s)(b/r) = (ab/sr)$. We notice that setting $(a/s) \geqslant 0$, if, and only if, $a \geqslant 0$, the ring (R, S) is an α-real ring. Moreover, it satisfies a somewhat stronger system of axioms stated below.

Let, R be a commutative ring with a relation $\geqslant 0$ as in the previous case. The β-axioms read as follows:

There is the identity e in R and

1β. *$a \geqslant 0$ and $b \geqslant 0$ implies $ab \geqslant 0$ and $a+b \geqslant 0$ and $a^{-1} \geqslant 0$ provided it exists*;
2β. *if $a \geqslant 0$, then $e+a$ is invertible*;
3β. *$a^2 \geqslant 0$ for each $a \in R$*;

---

*) We say a is thick, if $a \geqslant 0$ and $ax \geqslant 0$ implies $x \geqslant 0$ for each $x \in R$.

4β. *if* $e+na \geqslant 0$ *for* $n = 1, 2, \ldots$, *then* $a \geqslant 0$;
5β. $a \geqslant 0$ *and* $-a \geqslant 0$ *implies* $a = 0$.

THEOREM 1. *Every α-real ring R can be extended to a β-real ring* (R, S) *so that each* $x \in S$ *has an inverse in* (R, S). *The ring* (R, S) *is named the maximal extension of R.*

COROLLARY 1. (a/s) *is thick in* (R, S), *if and only if, a is thick in R.*

It can be proved that each β-real ring, is an α-real ring. Each element $a \geqslant 0$ of a β-real ring, which has an inverse, is thick. It follows directly from 5α. that if an element a is thick, then it cannot be a zero divisor. It can also be shown that thick elements of a β-real ring R can be characterized by the property that they are not zero divisors in the ring R', where R' is the completion of R with respect to the norm (*). Let R be the smallest β-real ring of functions on $\langle 0, 1\rangle$ with usual pointvise partial ordering on $\langle 0, 1\rangle$ and containing all the rational functions with poles outside $\langle 0, 1\rangle$ and the function $g(t) = \frac{1}{2}(\sin 2\pi t + |\sin 2\pi t|)$. There are no zero divisors in R and g is not thick. Let P be the ring of all rational functions with poles outside $\langle 0, 1\rangle$ and rational coefficients and with usual pointwise partial ordering on $\langle 0, 1\rangle$. An arbitrary not algebraic point outside $\langle 0, 1\rangle$ determines a homomorphism of P into reals which is not continuous with respect to (*).

COROLLARY 2. *By virtue of* 2β, *each β-real ring contains all the rational numbers* $m/n = m/ne$.

Let R be a β-real ring. We shall call $x \in R$ *bounded* if, and only if, there is a natural number n such that $x \leqslant ne$. It is clear that the set R_0, consisting of all bounded elements of R, is a β-real subring of R.

LEMMA 1. *Each element b of a β-real ring R can be represented as a quotient* $b = as^{-1}$, *where* $a, s \in R_0$ *and s is thick.*

Proof. It is sufficient to note that, for an arbitrary $b \in R$, the elements $b(e+b^2)^{-1}$ and $(e+b^2)^{-1}$ are bounded, and then, setting $a = b(e+b^2)^{-1}$, $s = (e+b^2)^{-1}$ we obtain $b = as^{-1}$, $a, s \in R$ and s is thick.

A homomorphism φ of R into the field of reals is *positive*, if and only if, $a \geqslant 0$ implies $\varphi(a) \geqslant 0$.

THEOREM 2. *If R is a β-real ring with all elements bounded, then each positive homomorphism is continuous with respect to* (*). *If, in addition, R is linear over the fields of reals, then each homomorphism into this field is positive.*

THEOREM 3. *Let R be a β-real ring that consists of bounded elements only, and let Φ be the set of all positive homomorphisms on R into the field of reals. The mapping*

$$\boldsymbol{F}(a) = F_a(\varphi) = \varphi(a) \qquad \varphi \in \Phi,$$

where $F_a(\varphi)$ is a function defined on Φ, is an isomorphic imbedding of R into the ring of bounded functions on Φ, with the property of $a \geqslant 0$ if, and only if, $\varphi(a) \geqslant 0$ for each $\varphi \in \Phi$.

This follows from

LEMMA 2. *If R is a β-real ring with all elements bounded, then the real function*

$$(*) \qquad \|a\| = \text{g. l. b } \{r : \pm a \leqslant r\},$$

where r is rational, has the properties:

1. $\|a+b\| \leqslant \|a\| + \|b\|$,
2. $\|ab\| \leqslant \|a\| \, \|b\|$,
3. $\|ra\| = |r| \, \|a\|$, *r rational,*
4. $\|a^2\| = \|a\|^2$,
5. $\|a\| = 0$ *if, and only if,* $a = 0$.

This, in turn, is a matter of simple calculation. For the sake of illustration we shall prove 2., and 4. We take any rational numbers r and s from the upper classes of Dedekind cuts of $\|a\|$ and $\|b\|$, and notice that $0 \leqslant (r+a)(s+b) = rs + rb + sa + ab$, $0 \leqslant (r-a)(s-b) = rs - rb - sa + ab$. Hence $0 \leqslant rs + ab$, $-ab \leqslant rs$ and since r and s are arbitrary $-ab \leqslant \|a\| \|b\|$. Similarily, the formulae $0 \leqslant (r-a)(s+b) = rs + rb - sa - ab$, $0 \leqslant (r+a)(s-b) = rs - rb + sa - ab$ yield $0 \leqslant rs - ab$, $ab \leqslant \|a\| \|b\|$, and, $as \|ab\|$ is the least number λ such that $\pm ab \leqslant \lambda$, property 2. follows. To prove 4. we notice first that

$$a^2 \leqslant e \quad \text{implies} \quad a \leqslant e.$$

In fact, $0 \leqslant e - a^2 = (e-a)(e+a) = (e-a)\big(2e-(e-a)\big) = 2(e-a) - (e-a)^2$, hence $2(e-a) \geqslant (e-a)^2 \geqslant 0$. Now it is easy to see that, for any rational r, $a^2 \leqslant r^2$ implies $a \leqslant r$, $\|a\| \leqslant r$, $\|a\|^2 \leqslant r^2$, and as the set of squares of rationals is dense 4. is established.

We sketch the proof of Theorem 3. The ring R can be treated as a metric space with the distance function $\varrho(a, b) = \|a-b\|$. We can make the completion of R to complete metric space R'. It is clear that the addition, multiplication and $\|a\|$ can be extended over R'. The extension of $\|a\|$ preserves properties 1., 2., and 5. Concerning 3., it holds even in a little stronger form $\|\lambda a\| = |\lambda| \, \|a\|$, where λ is an arbitrary real number. Therefore, $R'(\|a\|)$ is a commutative Banach algebra. By virtue of the Gelfand theorem [3] and 4. we see that $\lim_n \|a^n\|^{1/n} = \|a\|$. Hence, $a \in R \cap \cap \operatorname{Rad} R'$ implies that $a = 0$, and to each $a \neq 0$ we can find such a multiplicatively additive functional φ that $\varphi(a) \neq 0$ and $\varphi(x) \leqslant \|x\|$ for $x \in R'$. It is evident that the restriction of φ to the ring R is an element of Φ,

and so the mapping $\boldsymbol{F}(a) = F_a(\varphi) = \varphi(a)$ $\varphi \in \Phi$ is an isomorphism, and again by the theorem of Gelfand [3] $\|a\| =$ g. l. b. $\{r: \pm a \leqslant r\} = \sup\{|\varphi(a)|: \varphi \in \Phi\}$. It remains to be proved that $\varphi(a) \geqslant 0$ for each $\varphi \in \Phi$ if, and only if, $a \geqslant 0$. Let $\varphi(a) \geqslant 0$ and $a \leqslant r$. Then $0 \leqslant \varphi(r-a) = r - \varphi(a) \leqslant r$, and $\sup\{\varphi(r-a): \varphi \in \Phi\} = \|r-a\| \leqslant r$. Hence, $r - a \leqslant r + \frac{1}{n}$ for $n = 1, 2, \ldots$ and in virtue of 5β $a \geqslant 0$. The rest is clear.

We shall now examine the case of a β-real ring with some unbounded elements. Let R be an arbitrary β-real ring satisfying $1\beta - 5\beta$, and let T be an abstract set. Let us suppose that the subring R_0 which consists of bounded elements of R is a ring of real valued function on T such that $a \geqslant 0$ if, and only if, $a(t) \geqslant 0$ for each $t \in T$. We set

$$D_a = \bigcup_{a=bs^{-1}} \{t \in T: b, s \in R_0,\ s \geqslant 0,\ s(t) \neq 0\},$$

where $a \in R$ and all the s are thick. We can define the value of $a \in R$ on each $t \in D_a$ as follows: $a(t) = b(t)/s(t)$, where $a = b/s$, $s \geqslant 0$ and $s(t) \neq 0$.

THEOREM 4. 1. *$D_{ab} \cap D_{a+b} \supset D_a \cap D_b$ and $(a+b)t = a(t) + b(t)$, $(ab)t = a(t)\,b(t)$ on $D_a \cap D_b$. 2. If, for some $b \in R$, $a(t) \geqslant 0$ on $D_a \cap D_b$, then $a(t) \geqslant 0$ on D_a. 3. If $a(t) = 0$ on D_a, then $a = 0$.*

Theorem 4 leads to a natural definition of the real ring of functions. Let T be an abstract set and let $\mathcal{R}$ be a family of real valued functions defined on some subsets of T. D_a denotes the domain of $a \in \mathcal{R}$. The family $\mathcal{R}$ is said to be a *real ring* of *functions* if, and only if, the following conditions hold: 1. To every pair a, b of functions $\in \mathcal{R}$ there correspond some functions $ab, a+b, -a \in \mathcal{R}$, where $D_{ab} \cap D_{a+b} \supset D_a \cap D_b$, $D_{-a} = D_a$ and $(ab)t = a(t)\,b(t)$, $(a+b)t = a(t) + b(t)$ on $D_a \cap D_b$, $(-a)(t) = -a(t)$ on D_a. 2. If for some $b \in \mathcal{R}$, $a(t) \geqslant 0$ on $D_a \cap D_b$, then $a(t) \geqslant 0$ on D_a. 3. If for some $a, b \in \mathcal{R}$ we have $D_a \subset D_b$ and $a(t) = b(t)$ on D_a then $D_a = D_b$.

THE FIRST REPRESENTATION THEOREM. *Each a-real ring R can be isomorphically imbedded in a real ring of functions $\mathcal{R}$, in such a way that the non-negative elements are mapped on non-negative functions.*

Proof. Follows directly from Theorems 3 and 4.

We shall now introduce some topological aspects in our considerations. A β-real ring that consists of bounded elements only is said to be normal ([6], p. 223, def. 9) if, and only if, for each two elements $x, y \geqslant 0$ such that $x+y$ is invertible there exist $u, v \geqslant 0$ such that $x+u$ and $y+v$ are invertible, and $uv = 0$. An arbitrary β-real ring is normal if, and only if, the subring of bounded elements is normal.

THEOREM 5. *The space Φ considered in Theorem 3 with topology induced by the system of neighbourhoods*

$$(**) \qquad U(\varphi_0) = \{\varphi: |(\varphi - \varphi_0)a_i| < \varepsilon, \quad i = 1, 2, \ldots, n\}$$

is a compact Hausdorff space, and all the functions $F_a(\varphi) = \varphi(a)$ *are continuous in this topology.*

Let P be the set of all non-negative bounded elements of a β-real ring R. Clearly, P is a positive semi-ring ([6], p. 217, def. 2). We have the following

LEMMA 3. *To each maximal ideal* M *of* P *there corresponds a multiplicatively additive positive functional* φ *such that* $\{x \in P : \varphi(x) = 0\} = M$.

The set $\mathfrak{M}$ of maximal ideals of P is topologized, following Stone, by the family of neighbourhoods: $\Gamma_a = \{M : a \notin M\}$, which makes $\mathfrak{M}$ turn into T_1-space. By virtue of lemma 4 we can map Φ onto $\mathfrak{M}$ setting $g(\varphi) = M$, where $M = \{x \in P : y(x) = 0\}$. We have for $a \geqslant 0$ $g^{-1}(\Gamma_a) = \{\varphi : a \notin g(\varphi)\} = \{\varphi : \varphi(a) > 0\}$. This set is open in Φ, and so the mapping g is continuous. If R is normal, $\mathfrak{M}$ is a compact Hausdorff space ([6], p. 221, theorem 8) and the mapping is bicontinuous. Hence

THEOREM 6. *If* R *is normal, then the family of subsets* $\Gamma_a = \{\varphi : \varphi(a) > 0\}$, *where* a *runs over* P, *is a basis of neighbourhoods of* Φ *in the topology induced by* (**).

Now we turn back to the sets D_a of Theorem 4. In addition to the assumptions of this theorem, let the set T be a compact Hausdorff topological space and let all the functions of $\mathcal{R}$ be continuous on their domains. Clearly the sets D_a are open in T. Moreover, R being normal, we have

THEOREM 7. *The sets* D_a *are everywhere dense open subsets of* T.

Proof. It will be sufficient to show that each of the sets $\Gamma_s = \{t \in T : s \in \mathcal{R}_0,\ s \geqslant 0,\ s(t) \neq 0\}$ is everywhere dense in T whenever s has an inverse in R. But this is clear, since, if s has an inverse, $\Gamma_a \cap \Gamma_s = \Gamma_{as} \neq 0$ for $a \in P \cap R_0$, and by Theorem 6 Γ_s is everywhere dense.

Now, in turn, we shall make a topological specification of the real ring of functions defined above (cf. [7], p. 594). Let T be a compact Hausdorff space, and let $\mathcal{R}$ be a family of real valued functions defined on some dense subsets of T open everywhere. The family $\mathcal{R}$ is said to be a real ring of maximal continuous functions if, and only if:

1. to each $a, b \in \mathcal{R}$ there correspond $ab, a+b, -a \in \mathcal{R}$, such that $ab(t) = a(t)b(t)$, $(a+b)t = a(t)+b(t)$ on $D_a \cap D_b$, $(-a)t = -a(t)$ on D_a.

2. Every function is maximal in the sense that no function of R can be extended to a continuous function on a greater open subset than its own domain.

It is clear that each real ring of maximal continuous functions is a real ring of functions.

THE SECOND REPRESENTATION THEOREM. *Each normal β-real ring R can be isomorphically imbedded into a real ring of maximal continuous functions $\mathscr{R}$, so that $a \geqslant 0$ if, and only if, $F(a)$ is a non-negative function.*

This follows directly from Theorems 5 and 7.

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MATHEMATICS

On a Contractible Polytope which Cannot be Metrized in the Strong Convex Manner

by

K. SIEKLUCKI

Presented by K. BORSUK on March 29, 1958

A metric ϱ for the space X is said to be *strong convex* if for each two points: $a, b \in X$ there exists exactly one centre, i. e. such a point c that

$$\varrho(a, c) = \varrho(b, c) = \tfrac{1}{2}\varrho(a, b).$$

Each topological space X metrizable in the strong convex manner is contractible to a point, but this condition is not sufficient. There exist examples of spaces contractible to a point, admitting no strong convex metric, even among Absolute Retracts [1].

K. Borsuk has suggested that there also exist polytopes contractible to a point which cannot be metrized in the strong convex manner, and has pointed to the polytope B, constructed in his paper [2], as the eventual example.

This polytope is obtained from the interior and boundary of the circle with centre $(0, 0)$ and radius 1, by identification:

$(0, 0)$ with $(1, 0)$
$(x, 0)$ with $e^{2\pi i x}$ for $0 < x < 1$ (see Fig. 1).

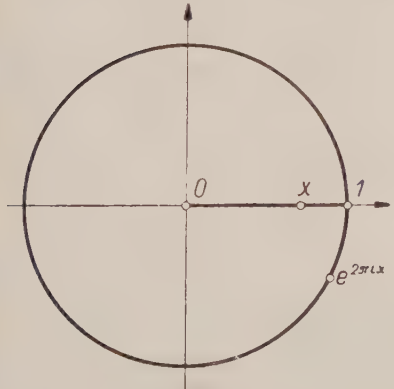

Fig. 1

Fig. 2

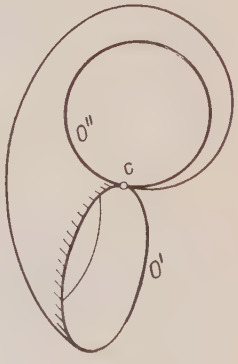

Fig. 3

We can also obtain B from the lateral surface of a cone (see Fig. 2) by matching: vertex c' with a point c'' belonging to basis 0, and segment $\overline{c', c''}$ with the basic circle 0.

Hence B is homeomorphic with the set of Fig. 3 in which circles $0'$ and $0''$ are identified.

Paper [2] contains the proof that B is contractible to a point.

The purpose of this paper is to prove that B is not metrizable in the strong convex manner.

LEMMA. *Let X be a compactum with a strong convex metric ϱ. If X contains at least two different points then for any given point $c \in X$ there exist a continuous function: $f: X \times I \to X$, and a point $p \in X$ satisfying the following conditions:*

1) $f(x, 0) = 0$, $f(x, 1) = c$ *for every* $x \in X$.

2) $f(x, t) \neq p$ *for every* $t \in (0, 1]$ *and every* $x \in X$.

Proof. It can be easily proved that in a strong convex complete space X, for each two points: $a, b \in X$ there exists exactly one segment $\overline{a, b}$ (i. e. an isometric image of the straight segment $[0, \varrho(a, b)]$) which depends continuously on a and b.

Given $c \in X$ we define:

$f(x, t) =$ the point on the segment $\overline{c, x}$, such that

$$\varrho(f(x, t), c) = (1-t) \cdot \varrho(x, c).$$

It is easy to see that condition 1) is satisfied.

Now we denote by p a point of X such that:

$$\varrho(p, c) = \sup_{x \in X} \varrho(x, c).$$

Suppose that for some $t_0 \neq 0$ and $x_0 \in X$ $f(x_0, t_0) = p$ holds. Then

$$\varrho(p, c) = \varrho(f(x_0, t_0), c) = (1-t_0) \cdot \varrho(x_0, c) < \varrho(x_0, c)$$

contrary to the definition of p. Hence, the condition 2) is also satisfied.

THEOREM. *The polytope B is not metrizable in the strong convex manner.*

Proof. Let us suppose that, in space B, there exists a strong convex metric ϱ.

Let $c \in B$ be the point obtained by matching of the points $(0, 0)$ and $(1, 0)$ (Fig. 1). Setting $X = B$ we infer that there exist a mapping $f: B \times I \to B$ and a point $p \in B$ satisfying our lemma.

Each point $x \in B - \{c\}$ has a neighbourhood V_x which is homeomorphic either with the interior of a circle (Fig. 4a) or with the Cartesian product of the union of three rays starting from one point, by an open segment (Fig. 4b).

Consequently, for each point $x \in B - \{c\}$ there exists the neighbourhood V_x which contains the subset V'_x satisfying the three following conditions:

1) $x \in V'_x$,

2) V'_x is homeomorphic with the interior of the circle,

3) $\mathrm{Fr}(V'_x)$ is the simple closed curve K_x.

It is shown in [2] that B may be imbeded in the 3-dimensional Euclidean space.

It is easy to observe that this imbedding can be chosen in such a manner that V'_p satisfies the condition:

4) There exists a straight line T, passing through p, such that, for some $\delta > 0$, $q \bar{\in} B$, if $q \in T$ and $0 < \varrho(p, q) \leqslant \delta$.

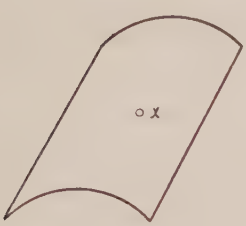

Fig. 4a

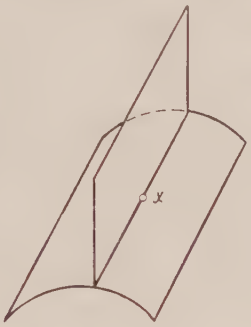

Fig. 4b

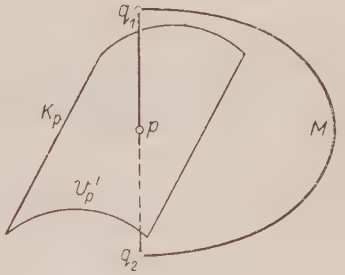

Fig. 5

Let q_1 and q_2 be two different points belonging to T and such that $\varrho(p, q_1) = \varrho(p, q_2) = \delta$.

It follows from condition 4) that

$$\overline{q_1, q_2} \cap B = \{p\}.$$

Let us observe that the space B as contractible to a point does not cut the 3-dimensional Euclidean space. Hence, the points q_1 and q_2 can be joined in $E_3 - (B \cap \overline{q_1, q_2})$ by the simple arc L. Thus we obtain a simple closed curve:

$$M = \overline{q_1, q_2} \cup L \qquad \text{(see Fig. 5).}$$

We see at once that

$$M \cap B = \{p\}. \tag{1}$$

Consider now the simple closed curve K_p bounding V'_p. Evidently, $p \bar{\in} K_p = f(K_p, 0)$. It follows by our lemma that

$$p \bar{\in} f(K_p, t) \quad \text{for every} \quad t \in [0, 1], \tag{2}$$

$$f(K_p, 1) = c. \tag{3}$$

In view of (1) and (2) we have:

$$M \cap f(K_p, t) = 0 \quad \text{for every} \quad t \in [0, 1]. \tag{4}$$

Evidently, the linking coefficient:

$$\mathfrak{d}(z^1_K, z^1_M) = \pm 1, \tag{5}$$

where z^1_k, z^1_M denote the orienting cycles of K_p and M. By (2) and (3) K_p is homotopic to zero in $B - \{p\}$, hence z^1_K is homologous to zero in $E_3 - M$ in contradiction with (5).

Thus, the proof of the theorem is completed.

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MATHEMATICS

On the Unions of Q-Spaces

by

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In [1] the author has shown that a normal space X which can be represented as the countable union of closed subsets, being Q-spaces in their relative topology, is again a Q-space. Moreover, in the same paper it was shown, by means of an example, that the assumption of normality of X is essential. In the present paper, we shall give *an example of a completely regular space X which can be represented as the union of two closed sets — each of which is a Q-space in its relative topology — which, however, is not a Q-space.*

Our space X will be built up with the use of the following space, which we denote as N: the space N consists of all points of the Euclidean plane lying on the line $y = 0$ and of all points lying above this line which has rational co-ordinates; the last points are isolated in N; neighbourhoods of a point p lying on the line $y = 0$ are of the form $K \cup \{p\}$, where K is the set of all points of N lying in an open disc tangential to the line $y = 0$ at a point p. Space N is a slight modification of a space constructed by Nemytzki, widely discussed in [3]. N is a Hausdorff, completely regular, non-normal space having everywhere dense, countable subset; moreover, the set of all points lying on the line $y = 0$ is a closed discrete subset of N of the power of the continuum. We shall denote this set as the basic set of N. It is shown in [2] that N is a Q-space.

The following terminology will be used: we say that a real-valued function f on a space S *separates a point $p_0 \in S$ from a set $A \subset S$ if $f(p_0) = 0$ and $f(p) > 0$ for each p in A*. It is known [1] that a completely regular space X is a Q-space if, and only if, each point of $\beta X / X$ can be separated from X by a continuous function on X (βX denotes the maximal Stone-Čech compactification of X).

I. We start with the following set-theoretic lemma:

LEMMA 1. *If R is a set of the power of the continuum and $\mathfrak{F}$ is a family of the power $\mathfrak{c}$ of functions on R, such that each member f of $\mathfrak{F}$ satisfies one of the following conditions*

(*) *f has continuously many values;*

(**) *there are at least two distinct elements α, β in $f(R)$ such that $\overline{\overline{f^{-1}(\alpha)}} = \overline{\overline{f^{-1}(\beta)}} = \mathfrak{c}$,*

*then there exists a permutation π *) of the set R, such that $f \cdot \pi \in \mathfrak{F}$ **) for each f in $\mathfrak{F}$.*

Proof. Let $\mathfrak{c} = \aleph_1$ and let

$$\{p_0, p_1, \ldots, p_\xi, \ldots\}_{\xi<\omega_\lambda} \tag{1}$$

be a sequence of all elements of R (we assume that $p_\xi \neq p_{\xi'}$ for $\xi \neq \xi'$), and let

$$\{f_0, f_1, \ldots, f_\xi, \ldots\}_{\xi<\omega_\lambda} \tag{2}$$

be a sequence of all members of $\mathfrak{F}$. In order to simplify the notations, we assume that if ξ is even ***), then f_ξ satisfies the condition (*), and if ξ is odd, then f_ξ satisfies the condition (**) (it is not necessary that each member of $\mathfrak{F}$ occurs only once in the sequence (2)). For each even ordinal $\xi < \omega_\lambda$ we denote as C_ξ a subset of R, such that for each α in $f_\xi(R)$, the intersection $f_\xi^{-1}(\alpha) \cap C_\xi$ consists of a single element. Of course, $\overline{\overline{C_\xi}} = \mathfrak{c}$. Moreover, for each odd ordinal $\xi < \omega_\lambda$ we denote by $C_\xi^{(0)}$ and $C_\xi^{(1)}$ two subsets of R of the power $\mathfrak{c}$ being of the form $C_\xi^{(i)} = f_\xi^{-1}(\alpha_i)$ $(i = 0, 1)$, where α_0 and α_1 are two distincts elements of $f_\xi(R)$. We define by the transfinite induction with respect to η the sets $K_\xi^{(\eta, i)}$ $(\xi, \eta < \omega_\lambda;\ i = 0, 1)$:

1° $K_\xi^{(0,i)} = 0$ for each $\xi < \omega_\lambda$ and $i = 0, 1$;

2° Suppose that, for some $\eta_0 < \omega_\lambda$, the sets $K_\xi^{(\eta, i)}$ are defined for each $\eta < \eta_0$, each $\xi < \omega_\lambda$ and $i = 0, 1$, and that they are of the power $< \mathfrak{c}$. Let us set $S = \bigcup \{K_\xi^{(\eta, i)} : \xi, \eta < \eta_0;\ i = 0, 1\}$. Let $K_0^{(\eta_0, 0)} = \bigcup \{K_0^{(\eta, 0)}; \eta < \eta_0\} \cup \{p\}$, where p is the first member of $C_0 \setminus S$ and $K_0^{(\eta_0, 1)} = 0$. Suppose that, for some $\xi_0 < \eta_0$, the sets $K_\xi^{(\eta_0, i)}$ are defined for each $\xi < \xi_0$ and that they are of the power $< \mathfrak{c}$. Let $S_{\xi_0} = \bigcup \{K_\xi^{(\eta_0, i)} : \xi < \xi_0;\ i = 0, 1\}$. If ξ_0 is even, then we put $K_{\xi_0}^{(\eta_0, 0)} = \bigcup \{K_{\xi_0}^{(\eta, 0)} : \eta < \eta_0\} \cup \{p\}$, where p is the first member of $C_{\xi_0} \setminus (S \cup S_{\xi_0})$, and $K_{\xi_0}^{(\eta_0, 1)} = 0$. If ξ_0 is odd, then we put $K_{\xi_0}^{(\eta_0, i)} = \bigcup \{K_{\xi_0}^{(\eta, i)} : \eta < \eta_0\} \cup \{p\}$, where p is the first member of $C_{\xi_0}^{(i)} \setminus (S \cup S_{\xi_0})$. Thus, the sets $K_\xi^{(\eta_0, i)}$ are defined for each $\xi < \eta_0$ and

*) i. e. π is a one-to-one mapping of R onto itself.

**) $f \cdot \pi$ denotes the superposition $f(\pi(p))$.

***) An ordinal ξ is called even if ξ can be represented in the form $\xi = \lambda + n$, where λ is a limit ordinal and n is an even integer, otherwise ξ is called odd.

$i = 0, 1$. Moreover, we put $K_\xi^{(\eta_0, i)} = 0$ for each $\xi \geqslant \eta_0$ and $i = 0, 1$. Thus, the sets $K_\xi^{(\eta_0, i)}$ are defined for each $\xi < \omega_\lambda$ and $i = 0, 1$. Hence, by transfinite induction, the sets $K_\xi^{(\eta, i)}$ are defined for each $\xi, \eta < \omega_\lambda$ and $i = 0, 1$.

Now, let

$$K_\xi^{(i)} = \bigcup \{K_\xi^{(\eta, i)} : \eta < \omega_\lambda\}.$$

It can easily be verified that the sets $K_\xi^{(i)}$ are mutually disjoint; moreover, if ξ is even, then $K_\xi^{(0)}$ is of the power $\mathfrak{c}$ and $K_\xi^{(0)} \subset C_\xi^{(0)}$; if ξ is odd, then $K_\xi^{(i)}$ are of the power $\mathfrak{c}$ and $K_\xi^{(i)} \subset C_\xi^{(i)}$ $(i = 0, 1)$.

Let $\xi < \omega_\lambda$ be any even ordinal. If π' and π'' are distinct permutations of $K_\xi^{(0)}$, then the functions $f \cdot \pi'$ and $f \cdot \pi''$ are also distinct. Since the set of all permutations of a set of the power $\mathfrak{c}$ is of the power $2^\mathfrak{c}$, one can select a permutation π_ξ of $K_\xi^{(0)}$, such that $f|K^{(0)} \neq f_\xi \cdot \pi_\xi$ *) for each f in $\mathfrak{F}$. Let $\xi < \omega_\lambda$ be any odd ordinal. If A is any subset of $K_\xi^{(0)}$ then a permutation π_A of the set $K_\xi^{(0)} \cup K_\xi^{(1)}$ may be selected, such that $\pi_A(p) \in K_\xi^{(1)}$ for each p in A and $\pi_A(p) = p$ for each p in $(K_\xi^{(0)} \cup K_\xi^{(1)}) \setminus (A \cup \pi_A(A))$; let $f_A = f \cdot \pi_A$. Of course, if A and B are distinct subsets of $K_\xi^{(0)}$, then the functions f_A and f_B are also distinct. It follows that a permutation π_ξ of the set $K_\xi^{(0)} \cup K_\xi^{(1)}$ may be selected, such that $f|K_\xi^{(0)} \cup K_\xi^{(1)} \neq f_\xi \cdot \pi_\xi$ for each f in $\mathfrak{F}$. Since the sets $K_\xi^{(i)}$ are mutually disjoint, there exists a permutation π of the set R which agrees with π_ξ on the domain of π_ξ for each $\xi < \omega_\lambda$. Of course, $f \cdot \pi \in \mathfrak{F}$ for each f in $\mathfrak{F}$ and the lemma follows.

It is plain that this lemma and its proof remain valid if we replace the number $\mathfrak{c}$ by any infinite regular cardinal $\mathfrak{m}$.

LEMMA 2. *Let R be a discrete subset of the power $\mathfrak{c}$ of a compact space Y. Then there exists a point $p_0 \in Y \setminus R$, such that, if g is a continuous function on Y, which separates p_0 from R, and $f = g|R$ is the restriction of g to the set R, then f satisfies either the condition* (*) *or the condition* (**).

Proof. The family $\mathfrak{A}$ of all subsets A of R for which $\overline{R \setminus A} < \mathfrak{c}$ is centred; hence there is p_0 in Y, such that $p_0 \in \bar{A}$ for each A in $\mathfrak{A}$. Of course, $p_0 \in Y \setminus R$. Now, let g be any continuous function on Y which separates p_0 from R. If $f = g|R$ satisfies neither (*) nor (**), then there exists a positive real α, such that $\overline{R \setminus f^{-1}(\alpha)} < \mathfrak{c}$, and it follows that $p_0 \in \overline{f^{-1}(\alpha)}$; hence $g(p_0) = \alpha > 0$ which leads to a contradiction.

II. Now we shall construct a space X which is not a Q-space, but can be represented as the union of two closed subsets being Q-spaces. Let N_1 and N_2 be two exemplars of the space N which was described

*) $f|K_\xi^{(0)}$ denotes the restriction of f to $K_\xi^{(0)}$.

at the beginning of the paper. We agree that points of N_2 are ordered pairs $\langle p, 0\rangle$, where $p \in N_1$. Let R_i be the basic set of the space N_i $(i = 1, 2)$. Of course, a function h defined on R_2 admits of a continuous extension over N_2 if, and only if, the function $f(p) = h(\langle p, 0\rangle)$ admits of a continuous extension over N_1. Denote as $\mathfrak{F}$ the family of all real-valued functions defined on R_1 which satisfies either the condition (*) or the condition (**) and which admits of a continuous extension over N_1. Then $\overline{\overline{\mathfrak{F}}} = \mathfrak{c}$; hence, by Lemma 1, there exists a permutation π of the set R_1, such that $f \cdot \pi \bar{\in} \mathfrak{F}$ for each f in $\mathfrak{F}$ (i. e. $f \cdot \pi$ admits of no continuous extension over N_1). Now, let X be the space obtained from N_1 and N_2 by identifying each point $p \in R_1$ with the point $\langle \pi^{-1}(p), 0\rangle \in R_2$. Of course, this identification leads to a completely regular Hausdorff space. Moreover, X contains N_1 and N_2 as closed subsets; thus X can be represented as the union of two closed subsets being Q-spaces.

We shall show that X is not a Q-space. Denote as Y the closure of R_1 with respect to βX and let p_0 be a point of $Y \setminus R_1$ whose existence asserts Lemma 2. Suppose that g is a continuous function on βX which separates p_0 from X. Denote as f_i the restriction of g to the set R_i $(i = 1, 2)$. By Lemma 2, f_1 satisfies the condition (*) or (**). Moreover, f_1 admits of a continuous extension over N_1; hence $f_1 \in F$. But f_2 satisfies the equality $f_2(\langle \pi^{-1}(p), 0\rangle) = f_1(p)$ i. e. $f_2(\langle p, 0\rangle) = f_1(\pi(p))$. Thus $f_1 \cdot \pi \bar{\in} \mathfrak{F}$, hence f_2 admits no continuous extension over N_2 and it leads to a contradiction. This shows that p_0 cannot be separated from X by a continuous function on βX, and that X it not a Q; space.

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MATHÉMATIQUE

Propriétés des solutions du système parabolique d'équations aux dérivées partielles

par

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Soit le système de $N \geqslant 1$ équations aux dérivées partielles d'ordre $M \geqslant 2$ de la forme

$$(1)\quad \hat{\Psi}^{(\alpha)}(u_1, \ldots, u_n) = \sum_{1\leqslant j\leqslant N}^{k_1+\ldots+k_n=M} A_{\alpha j}^{k_1\ldots k_n}(X,t)\frac{\partial^M u_j}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}} + \sum_{1\leqslant j\leqslant N}^{1\leqslant k_1+\ldots+k_n<M} B_{\alpha j}^{k_1\ldots k_n}(X,t)\frac{\partial^{k_1+\ldots+k_n}u_j}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}} + \sum_{1\leqslant j\leqslant N} C_{\alpha j}(X,t)u_j - \frac{\partial u_\alpha}{\partial t} = 0$$

$(\alpha = 1, 2, \ldots N)$, où $u_1(X,t), \ldots u_N(X,t)$ sont des fonctions inconnues dans la région

$$(2)\qquad X \in \Omega; \quad 0 \leqslant t \leqslant T.$$

Ω désigne un domaine mesurable dans l'espace euclidien E des points $X(x_1, \ldots x_n)$.

Nous admettons pour les coefficients $A_{\alpha j}$, $B_{\alpha j}$, $C_{\alpha j}$ les mêmes hypothèses que dans notre travail [1]. Dans ce travail nous avons construit les N solutions fondamentales du système (1) formées par les éléments

$$\Gamma_{1\beta}, \Gamma_{2\beta}, \ldots \Gamma_{N\beta}; \qquad (\beta = 1, 2, \ldots N)$$

de la matrice déterminée par les formules

$$(3)\quad \Gamma_{\alpha\beta}(X,t;Y,\tau) = W_{\alpha\beta}^{Y,\tau}(X,t;Y,\tau) + \int_\tau^t \iiint_E \sum_{\gamma=1}^N W_{\alpha\gamma}^{\Pi,\zeta}(X,t;\Pi,\zeta)\Phi_{\gamma\beta}(\Pi,\zeta;Y,\tau)\,d\Pi\,d\zeta,$$

où $X(x_1, \ldots x_n) \neq Y(\xi_1, \ldots \xi_n)$ sont les deux points arbitraires de l'espace E et $0 < \tau < t < T$.

Les fonctions $W_{\alpha\beta}^{Z,\zeta}(X, t; Y, \tau)$, dites composantes des quasi-solutions, sont déterminées par les formules (7) et (8) de notre travail [1]. Les fonctions $\Phi_{\alpha\beta}(X, t; Y, \tau)$ forment les solutions du système d'équations de Volterra (18) du travail [1], vérifiant les limitations (19) et (19′).

Dans cette communication nous présenterons le résultat de nos recherches, concernant certaines solutions du système (1) formées par les solutions fondamentales (3).

THÉORÈME 1. *Si les fonctions* $\varrho_\beta(Y, \tau)$ *sont définies et continues dans la région fermée et bornée*

$$Y \in \Omega; \quad 0 \leqslant \tau \leqslant T, \tag{4}$$

les fonctions *) *(analogues à l'intégrale de Poisson-Weierstrass)*

$$J_\alpha(X, t, \tau) = \iiint_{\Omega} \sum_{\beta=1}^{N} \Gamma_{\alpha\beta}(X, t; Y, \tau)\, \varrho_\beta(Y, \tau)\, dY \tag{5}$$

vérifient le système d'équations

$$\hat{\Psi}^{(\alpha)}[J_1, J_2, \ldots J_N] = 0; \quad (\alpha = 1, 2, \ldots N) \tag{6}$$

en tout point X de l'espace E, pour $0 \leqslant \tau < t < T$, et tendent vers les limites

$$\lim_{\tau \to t} J_\alpha(X, t, \tau) = \varrho_\alpha(X, t) \tag{7}$$

uniformément par rapport à X en tout domaine fermé Ω^* situé à l'intérieur du domaine donné et mesurable $\Omega \subset E$.

Définition. On appelle potentiel de charge spatiale, relatif au système (1), une suite de N fonctions $\{V_\alpha(X, t)\}$ données par les intégrales de volume

$$V_\alpha(X, t) = \int_0^t \iiint_{\Omega} \sum_{\beta=1}^{N} \Gamma_{\alpha\beta}(X, t; Y, \tau)\, \varrho_\beta(Y, \tau)\, dY\, d\tau \tag{8}$$

$(\alpha = 1, \ldots N)$, où les N fonctions de la suite $\{\varrho_\beta(Y, \tau)\}$, dite densité de la charge, sont déterminées, bornées et intégrables dans la région (4).

THÉORÈME 2. *Si les composantes* $\varrho_\beta(Y, \tau)$ *de la densité sont bornées et intégrables dans la région* (4), *les composantes* (8) *du potentiel de charge spatiale admettent les dérivées spatiales d'ordre* $m \leqslant M-1$ *continues en tout point de l'espace E, données par les intégrales absolument convergentes*

$$D_X^{(m)}[V_\alpha(X, t)] = \int_0^t \iiint_{\Omega} \sum_{\beta=1}^{N} D_X^{(m)}[\Gamma_{\alpha\beta}(X, t; Y, \tau)]\, \varrho_\beta(Y, \tau)\, dY\, d\tau, \tag{9}$$

*) Nous désignons dans ce travail par le symbole de l'intégrale triple celle de volume dans l'espace E à n dimensions; dY désigne l'élément de volume au point Y.

qui vérifient les inégalités

$$|D_X^{(m)}[V_\alpha(X,t)]| < \text{const} \cdot t^{1-\mu} \sup |\varrho_\beta| , \tag{10}$$

où $\sup|\varrho_\beta|$ *désigne la plus grande des bornes supérieures des fonctions* $|\varrho_\beta|$, μ *est choisi arbitrairement à l'intérieur de l'intervalle* $\left(\frac{m}{M}, 1\right)$.

THÉORÈME 3. *Si les fonctions de la densité* $\{\varrho_\beta(Y,\tau)\}$ *sont bornées et intégrables dans le domaine* (4), *les composantes* (8) *du potentiel de charge spatiale et leurs dérivées spatiales d'ordre* $m < M$ *vérifient dans le domaine* (4) *les conditions d'Hölder relativement à la variable* t *des formes suivantes:*

$$|V_\alpha(X,t) - V_\alpha(X,t_1)| < \text{const.} \sup|\varrho_\beta| \cdot |t-t_1|^\theta \tag{11}$$

$$|D_X^{(m)}[V_\alpha(X,t)] - D_X^{(m)}[V_\alpha(X,t)]| < \text{const.} \sup|\varrho_\beta| \cdot |t-t_1|^{\theta\left(1-\frac{m}{M}\right)} \tag{12}$$

$(m = 1, 2, \ldots M-1;\ \alpha = 1, \ldots N)$, θ *désignant une constante positive arbitraire, inférieure à l'unité; les coefficients constants positifs ne dépendent pas des fonctions* ϱ_β.

THÉORÈME 4. *Sous les mêmes hypothèses que dans le théorème* 3, *les dérivées spatiales d'ordre* $M-1$ *des composantes* (8) *du potentiel de charge spatiale vérifient dans le domaine* (4) *la condition d'Hölder par rapport aux variables spatiales*

$$|D_X^{(M-1)}[V_\alpha(X,t)] - D_X^{(M-1)}[V_\alpha(X_1,t)]| < \text{const.} \sup|\varrho_\beta| \cdot |XX_1|^\theta , \tag{13}$$

θ *désignant une constante positive arbitraire, inférieure à l'unité,* $|XX_1|$ *désignant la distance euclidienne entre les points* X *et* X_1 *du domaine* Ω.

THÉORÈME AUXILIAIRE. *Si les fonctions* $\varrho_\beta(Y,\tau)$ *sont bornées et intégrables dans la région* (4), *les fonctions*

$$\bar{\varrho}_\gamma(\Pi,\zeta) = \int_0^\zeta \iiint_\Omega \sum_{\beta=1}^N \Phi_{\gamma\beta}(\Pi,\zeta;\ Y,\tau)\,\varrho_\beta(Y,\tau)\,dY\,d\tau \tag{14}$$

vérifient dans cette région la condition d'Hölder

$$|\bar{\varrho}_\gamma(\Pi,\zeta) - \bar{\varrho}_\gamma(\Pi_1,\zeta)| < \text{const.} \sup|\varrho_\beta| \cdot |\Pi\Pi_1|^{\theta h_1} \tag{15}$$

où $h_1 = \min(h, 2h')$, θ *est une constante positive arbitraire, inférieure à l'unité.*

THÉORÈME 5. *Si les composantes de la densité* $\{\varrho_\beta(Y,\tau)\}$ *vérifient dans la région* (4) *la condition d'Hölder par rapport aux variables spatiales*

$$|\varrho_\beta(Y,\tau) - \varrho_\beta(Y_1,\tau)| < \text{const} \cdot |YY_1|^{h_\varrho} \tag{16}$$

$(0 < h_\varrho \leqslant 1)$ *et sont continues par rapport à la variable* t, *alors les composantes du potentiel de charge spatiale* (8) *admettent les dérivées spatiales*

d'ordre M, en tout point intérieur $X \in \Omega$, pour $0 < t < T$, données par les formules

$$(17)\quad D_X^{(M)}[V_\alpha(X,t)] = \int_0^t \iint_{\Omega(Y)} \sum_{\beta=1}^{N} D_X^{(M)}[W_{\alpha\beta}^{Y,\tau}(X,t;\,Y,\tau)]\,\varrho_\beta(Y,\tau)\,dY\,d\tau + \\ + \int_0^t \iint_{E(\Pi)} \sum_{\gamma=1}^{N} D_X^{(M)}[W_{\alpha\gamma}^{\Pi,\zeta}(X,t;\,\Pi,\zeta)]\,\bar{\varrho}_\gamma(\Pi,\zeta)\,d\Pi\,d\zeta\,,$$

où les intégrales ne sont pas, en général, absolument convergentes.

THÉORÈME 6. *Si les conditions du théorème 5 sont satisfaites, les composantes du potentiel (8) admettent les dérivées par rapport à la variable t données par la formule*

$$(18)\quad \frac{\partial}{\partial t}[V_\alpha(X,t)] = \varrho_\alpha(X,t) + \\ + \int_0^t \iint_{\Omega} \sum_{\beta=1}^{N} \frac{\partial}{\partial t}[W_{\alpha\beta}^{Y,\zeta}(X,t;\,Y,\tau)]\,\varrho_\beta(Y,\tau)\,dY\,d\tau + \\ + \bar{\varrho}_\alpha(X,t) + \int_0^t \iint_{E} \sum_{\gamma=1}^{N} \frac{\partial}{\partial t}[W_{\alpha\gamma}^{\Pi,\zeta}(X,t;\,\Pi,\zeta)]\,\bar{\varrho}_\gamma(\Pi,\zeta)\,d\Pi\,d\zeta\,,$$

en tout point intérieur $X \in \Omega$, pour $0 < t < T$.

THÉORÈME 7. *Si les conditions du théorème 5 sont satisfaites, le potentiel de charge spatiale $\{V_\alpha(X,t)\}$, donné par les formules (8), vérifie le système d'équations*

$$(19)\quad \hat{\Psi}^{(\alpha)}[V_1(X,t), \ldots V_N(X,t)] = -\varrho_\alpha(X,t)$$

$(\alpha = 1, 2, \ldots N)$ en tout point intérieur $X \in \Omega$ pour $0 < t < T$; $\hat{\Psi}^{(\alpha)}$ désignent les opérations différentielles, définies par les formules (1).

THÉORÈME 8. *Si la densité continue $\{\varrho_\beta(Y,\tau)\}$ vérifie la condition d'Hölder*

$$(20)\quad |\varrho_\beta(Y,\tau) - \varrho_\beta(Y_1,\tau)| < \varkappa_\varrho |Y Y_1|^{h_\varrho}$$

$(0 < h_\varrho \leqslant 1)$ dans le domaine borné Ω pour $0 \leqslant \tau \leqslant T$, alors les dérivées spatiales d'ordre M des composantes (8) du potentiel de charge spatiale vérifient la condition d'Hölder de la forme

$$(21)\quad |D_X^{(M)}[V_\alpha(X,t)] - D_X^{(M)}[V_\alpha(X_1,t_1)]| < \\ < (C_1 \sup|\varrho_\beta| + C_2 \varkappa_\varrho)[|XX_1|^{h_*} + |t - t_1|^{\frac{h}{h+M}\theta'}]$$

dans tout domaine fermé Ω^* situé à l'intérieur du domaine Ω, $[t, t_1 \in (0, T)]$, où $h_* = h_\varrho$, si $h_\varrho < h_1 = \min(h, 2h')$ et $h_* = \theta h_1$, si $h_\varrho \geqslant h_1$; θ et θ' sont des constantes positives arbitraires, inférieures à l'unité; C_1 et C_2 sont des constantes positives dépendant du domaine Ω^*.

THÉORÈME 9. *Si les fonctions* $F_\alpha(X, t)$ $(\alpha = 1, 2, \dots N)$ *sont définies et continues dans la région*

$$X(x_1, \dots x_n) \in E; \quad 0 \leqslant t \leqslant T \tag{22}$$

et vérifient la condition d'Hölder par rapport aux variables spatiales

$$|F_\alpha(X, t) - F_\alpha(X_1, t)| < \text{const} \cdot |XX_1|^{h_F} \tag{23}$$

(pour $|XX_1|$ *est inférieure à un nombre positif fixé), si les fonctions* $f_\alpha(X)$ *sont définies et continues dans tout l'espace* E, *si en outre ces fonctions vérifient les inégalités*

$$\begin{aligned} |F_\alpha| &< A|XX_0|^p + B, \\ |f_\alpha| &< A|XX_0|^p + B \end{aligned} \tag{24}$$

(A, B, p *étant des constantes positives et* X_0 *— un point fixé), alors les formules*

$$u_\alpha(X, t) = -\int_0^t \iiint_E \sum_{\beta=1}^N \Gamma_{\alpha\beta}(X, t; Y, \tau) F_\beta(Y, \tau)\, dY\, d\tau + \tag{25}$$

$$+ \iiint_E \sum_{\beta=1}^N \Gamma_{\alpha\beta}(X, t; Y, 0) f_\beta(Y)\, dY$$

$(\alpha = 1, 2, \dots N)$ *présentent une solution* $(u_1, u_2, \dots u_N)$ *du problème de Cauchy pour le système d'équations aux dérivées partielles*

$$\hat{\Psi}^{(\alpha)}(u_1, u_2, \dots u_N) = F_\alpha(X, t)\,; \tag{26}$$

$(\alpha = 1, 2, \dots N)$ *c. à d. les fonctions* (25) *vérifient les équations* (26) *en tout point* X *de l'espace* E *pour* $0 < t < T$ *et ces fonctions satisfont aux conditions initiales*

$$\lim_{t \to 0} u_\alpha(X, t) = f_\alpha(X) \tag{27}$$

en tout point $X \in E$.

Les démonstrations des théorèmes 1-9 seront publiées dans un travail qui paraîtra prochaînement.

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MATHÉMATIQUE

Sur le plongement k-indépendant d'espaces euclidiens

par

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Présenté par K. BORSUK le 18 avril, 1958

Cette communication a pour but de résoudre un problème posé par K. Borsuk (voir [1], p. 355).

Soit E_α l'espace euclidien à α dimensions lorsque α est un nombre ordinal fini et l'espace de Hilbert lorsque $\alpha = \omega$. Appelons avec Borsuk ([1], p. 351) *k-indépendant* tout sous-ensemble A de E_α tel que n'importe quels $i \leqslant k+2$ points de A (k étant un nombre naturel) sont linéairement indépendants; si A est k-indépendant pour tout $k = 1, 2, \ldots$, il est dit *ω-indépendant.*

X étant un espace métrique séparable, désignons par $D_k(X)$, où $k \leqslant \omega$, le plus petit $\alpha \leqslant \omega$, tel que X est homéomorphe à un sous-ensemble k-indépendant de E_α (cf. [1], p. 354). Un tel $D_k(X)$ existe, d'où en particulier $D_\omega(E_\alpha) \leqslant \omega$ pour tout $\alpha \leqslant \omega$ ([1], p. 351, Theorem) et l'on a pour tout $k = 1, 2, \ldots$

$$D_k(E_1) \leqslant k+1, \tag{1}$$

$$D_k(E_2) \leqslant 2k+2. \tag{2}$$

Le problème dont il s'agit est le suivant: le nombre $D_k(E_n)$ est-il fini pour tout couple k, n de nombres finis? La réponse est affirmative.

THÉORÈME. *On a* $D_k(E_n) \leqslant \frac{1}{2}(k+1)[(k+2)(n-1)+2]$ *pour tout* $k = 1, 2, \ldots$ *et tout* $n = 1, 2, \ldots$

Démonstration. R étant le corps des nombres réels, considérons les systèmes finis

$$(x_0, x_1, \ldots, x_r) \quad \text{où} \quad x_i \in R \quad \text{pour} \quad i = 0, 1, \ldots, r$$

et appelons deux systèmes

$$(x_0, x_1, \ldots, x_r) \quad \text{et} \quad (y_0, y_1, \ldots, y_s) \quad \text{où} \quad r \leqslant s \tag{3}$$

équivalents lorsqu'on a simultanément

$$x_i = y_i \quad \text{pour} \quad i = 0, 1, \ldots, r,$$
$$y_i = 0 \quad \text{pour} \quad i = r+1, \quad r+2, \ldots, s.$$

L'ensemble A des classes d'abstraction de cette équivalence devient un anneau en y définissant l'addition et la multiplication des représentants (3) de ces classes comme il suit:

$$(4) \qquad (x_0, x_1, \ldots, x_r) + (y_0, y_1, \ldots, y_s) = (z_0, z_1, \ldots, z_s)$$

où $z_i = x_i + y_i$ pour $i = 0, 1, \ldots, s$ et $x_i = 0$ pour $i = r+1, r+2, \ldots, s$;

$$(5) \qquad (x_0, x_1, \ldots, x_r) \cdot (y_0, y_1, \ldots, y_s) = (z_0, z, \ldots, z_{r+s})$$

où $z_i = x_i \cdot y_0 + x_{i-1} \cdot y_1 + \ldots + x_0 \cdot y_i$ pour $i = 0, 1, \ldots, r+s$, $x_i = 0$ pour $i = r+1, r+2, \ldots, r+s$ et $y_i = 0$ pour $i = s+1, s+2, \ldots, r+s$.

Les deux opérations ainsi définies entre classes sont indépendantes du choix des éléments respectifs (3) de ces classes. On peut donc identifier dans l'écriture $a \in A$ avec $(x_0, x_1, \ldots, x_r) \in a$. Il est aisé de voir que l'anneau A est sans diviseurs du zéro, avec une unité et que tout $(x_0, x_1, \ldots, x_r)$ a un développement unique, à savoir

$$(6) \qquad (x_0, x_1, \ldots, x_r) = x_0 \cdot e_0 + x_1 \cdot e_1 + \ldots + x_r \cdot e_r$$

où $e_i = (\underbrace{0, \ldots, 0}_{i \text{ fois}}, 1)$ pour $i = 0, 1, \ldots, r$ *).

Soit f la transformation de E_n en un sous-ensemble $S = f(E_n)$ de A, faisant correspondre à tout point $(x_0^{(i)}, x_1^{(i)}, \ldots, x_{n-1}^{(i)})$ de E_n l'élément $a_i = (x_0^{(i)}, x_1^{(i)}, \ldots, x_{n-1}^{(i)})$ de S.

Soit g la transformation de S en un sous-ensemble $T = g(S)$ du produit cartésien $\underbrace{A \times A \times \ldots \times A}_{k+1 \text{ fois}}$ faisant correspondre à tout élément a_i de S l'élément $(a_i, a_i^2, \ldots, a_i^{k+1})$ de T (où $a_i^j = \underbrace{a_i \cdot a_i \cdot \ldots \cdot a_i}_{j \text{ fois}}$).

Quels que soient $k+2$ éléments $a_1, a_2, \ldots, a_{k+2}$ de T, la matrice

$$(7) \qquad \begin{pmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{k+1} \\ 1 & a_2 & a_2^2 & \ldots & a_2^{k+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & a_{k+2} & a_{k+2}^2 & \ldots & a_{k+2}^{k+1} \end{pmatrix}$$

est de rang $k+2$, car c'est une matrice de Vandermonde.

*) L'anneau A est d'ailleurs isomorphe à celui des polynômes aux coefficients réels à une variable. La définition de A donnée dans le texte n'a été choisie qu'en vue de rendre plu claire l'idée de la démonstration.

Enfin, en posant pour abréger

$$\varphi(k, n) = \tfrac{1}{2}(k+1)[(k+2)(n-1)+2] \tag{8}$$

(cf. l'énoncé du théorème), une transformation h de T en un sous-ensemble $U = h(T)$ de E_m, où $m \geqslant \varphi(k, n)$ est arbitrairement donné d'avance, va être définie, qui fera correspondre à tout élément $(a, a^2, \ldots, a^{k+1})$ de T l'élément $(x_0, x_1, \ldots, x_n, \ldots, x_{\varphi(k,n)})$ de U, déterminé comme il suit:

Tout terme de la matrice (7) est en vertu de (6) de la forme

$$a_i^j = y_{0,j}^{(i)} \cdot e_0 + y_{1,j}^{(i)} \cdot e_1 + \ldots + y_{n-1,j}^{(i)} \cdot e_{n-1} \tag{9}$$

où $i = 1, 2, \ldots, k+2$ et $j = 0, 1, \ldots, k+1$. En ajoutant à la matrice (7) des colonnes supplémentaires convenables, on peut obtenir de (7) par des opérations élémentaires la matrice

$$\begin{pmatrix} 1 & y_{0,1}^{(1)} & \ldots & y_{n-1,1}^{(1)} & y_{0,2}^{(1)} & \ldots & y_{2(n-1),2}^{(1)} & \ldots & y_{0,k+1}^{(1)} & \ldots & y_{(k+1)(n-1),k+1}^{(1)} \\ 1 & y_{0,1}^{(2)} & \ldots & y_{n-1,1}^{(2)} & y_{0,2}^{(2)} & \ldots & y_{2(n-1),2}^{(2)} & \ldots & y_{0,k+1}^{(2)} & \ldots & y_{(k+1)(n-1),k+1}^{(2)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & y_{0,1}^{(k+2)} & \ldots & y_{n-1,1}^{(k+2)} & y_{0,2}^{(k+2)} & \ldots & y_{2(n-1),2}^{(k+2)} & \ldots & y_{0,k+1}^{(k+2)} & \ldots & y_{(k+1)(n-1),k+1}^{(k+2)} \end{pmatrix} \tag{10}$$

où $y_{p,1}^{(i)} = x_q^{(i)}$ pour $q = 0, 1, \ldots, n-1$ et $y_{p,j}^{(i)}$ pour $j = 2, 3, \ldots, k+1$ sont des fonctions des mêmes variables $x_q^{(i)}$ de la forme

$$y_{p,j}^{(i)} = \sum x_{u_1}^{(i)} \cdot x_{u_2}^{(i)} \cdot \ldots \cdot x_{u_j}^{(i)},$$

où

$$u_1 + u_2 + \ldots + u_j = p; \quad p = 0, 1, \ldots, (k+1)(n-1) \quad \text{et} \quad i = 1, 2, \ldots, k+2.$$

Il y a $\varphi(k, n)+1$ termes dans toute ligne de la matrice (10) et son rang est aussi égal à $k+2$, car s'il était plus petit, on pourrait multiplier tout terme $y_{p,j}^{(i)}$ par l'unité e_p et remplacer chacun des produits $y_{0,j}^{(i)} \cdot e_0$ par la somme (9), c'est-à-dire par a_i^j, en obtenant ainsi une matrice qui ne diffère de (7) que par la présence de certaines colonnes supplémentaires, donc ayant au moins le rang égal au sien, qui est $k+2$.

En désignant par $x_n^{(i)}$ le n-ième terme de la i-ème ligne de la matrice (10), on peut représenter cette matrice dans la forme

$$\begin{pmatrix} 1 & x_0^{(1)} & x_1^{(1)} & \ldots & x_n^{(1)} & \ldots & x_{\varphi(k,n)}^{(1)} \\ 1 & x_0^{(2)} & x_1^{(2)} & \ldots & x_n^{(2)} & \ldots & x_{\varphi(k,n)}^{(2)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x_0^{(k+2)} & x_1^{(k+2)} & \ldots & x_n^{(k+2)} & \ldots & x_{\varphi(k,n)}^{(k+2)} \end{pmatrix}. \tag{11}$$

Si l'on fait donc correspondre à tout élément de T composé de termes figurant dans une même ligne de la matrice (7) le point de l'espace $E_{\varphi(k,n)}$ ayant pour coordonnées les termes consécutifs de la même ligne de la matrice (11), la transformation h se trouve définie.

La superposition des transformations f, g et h donne une homéomorphie

$$hgf(E_n) = U \subset E_m \quad \text{où} \quad m \geqslant \varphi(k, n),$$

car n premières coordonnées de l'image (point de E_m) coïncident avec celles de son modèle (point de E_n) et les autres $m-n$ coordonnées en sont des fonctions bicontinues (d'ailleurs pas nécessairement biunivoques). En même temps, la matrice (11) écrite pour les images de $k+2$ points quelconques de E_n

$$hgf((x_0^{(i)}, x_1^{(i)}, \ldots, x_{n-1}^{(i)})) = (x_0^{(i)}, x_1^{(i)}, \ldots, x_n^{(i)}, \ldots, x_{(k,n)}^{(i)}),$$

où $i = 1, 2, \ldots, k+2$, étant de rang $k+2$, ces images sont des points linéairement indépendants. Le plongement est donc bien k-indépendant. Enfin, l'image homéomorphe U de E_n étant située, comme on vient de voir, déjà dans $E_{\varphi(k,n)}$ et $D_k(E_n)$ étant par définition le plus petit nombre de dimensions d'un espace euclidien dans lequel E_n se laisse plonger ainsi par homéomorphie, on a $D_k(E_n) \leqslant \varphi(k, n)$, ce qui achève la démonstration en vertu de (8).

Corollaire. *Si X est un espace métrique séparable et* $\dim(X) \leqslant n$, *on a* $D_k(X) \leqslant (k+1)[(k+2)n+1]$.

C'est le résultat direct de la substitution de la valeur de $\varphi(k, 2n+1)$ d'après (8) dans l'énoncé (16) de [1], p. 355.

Remarques. Le théorème qui vient d'être établi donne pour $n = 1$ la même estimation que (1) et pour $n = 2$ une estimation plus faible que (2); mais l'estimation $D_k(E_n) \leqslant \varphi(k, n)$ fournie par lui est générale. L'évaluation de la valeur exacte de $D_k(E_n)$ reste un problème ouvert.

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OUVRAGES CITÉS

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MATHEMATICS

A Remark About Cartesian Division by a Segment

by

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Presented by K. BORSUK on April 16, 1958

Let A be the complement of a single point in Euclidean n-space E^n ($n \geqslant 2$), B an open subset of E^n and I an open interval of real numbers. Professor A. Bielecki has raised the question, *whether the homeomorphism of Cartesian products $A \times I$ and $B \times I$ does imply the homeomorphism of A and B.* The aim of this note is to prove that the answer is negative if $n \geqslant 3$. In the case $n = 3$ we produce a counter-example based on the known paper of Fox and Artin [1] and our construction can easily be extended to higher dimensions.

In a spherical n-space S^n, a curved polyhedron P is tamely imbedded [1] if there exists a homeomorphism of S^n on itself which maps P on a Euclidean polyhedron. If no such homeomorphism exists, P is wildly imbedded. Let us restate briefly the construction and some properties of the wild arc Y of [1; 1.2]. Let C be a right circular cylinder containing three disjoint, oriented, simple polygonal arcs K_-, K_0 and K_+. These three arcs, which have only their end-points in common with the boundary of C, are arranged as indicated in Fig. 1.

Let $K = K_- \cup K_0 \cup K_+$. The family of parallel planes $x = 2 - 2^{1-m}$

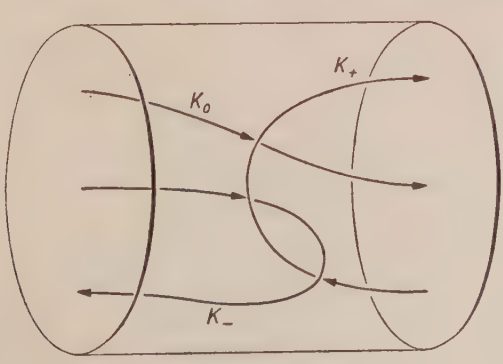

Fig. 1

$m = 0, 1, 2, \ldots$, divides the half-ellipsoid of revolution $E: x^2 + 4(y^2 + z^2) \leqslant 4$, $x \geqslant 0$, into an infinite number of sections D_n defined by inequalities $2 - 2^{2-n} \leqslant x \leqslant 2 - 2^{1-n}$, $n = 1, 2, 3, \ldots$ Denote by q the vertex $(2, 0, 0)$. For each positive integer n we choose such a homeomorphism f_n of C on D_n, that the set

$$Y = f_1(K_0 \cup K_+) \cup \bigcup_{n=2}^{\infty} f_n(K) \cup q$$

is an infinite, simple polygonal arc having only end-points $o = (0, 0, 0)$ and q in common with the boundary of E, shown in Fig. 2.

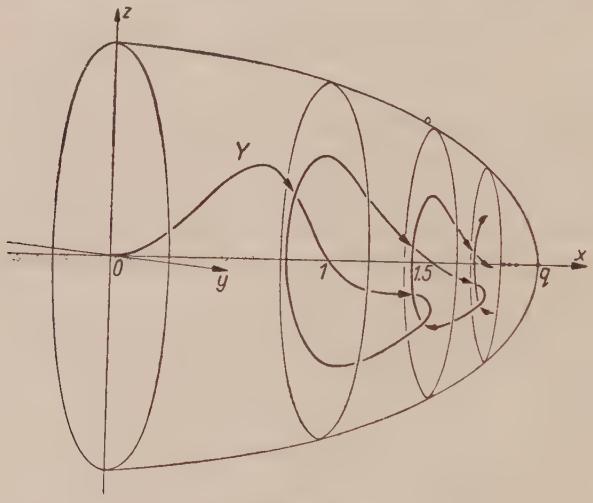

Fig. 2

By "thickening" slightly the edges we get an infinite polyhedral 3-cell E^* whose interior contains $Y - q$. There is such a mapping g of S^3 on itself that $g|_{\overline{S^3 - E^*}}$ is the identity mapping, g maps topologically the interior of $E^* - Y$ on the interior of E^* and $g(Y) = q$. So the complement $S^3 - Y$ of Y is an open 3-cell.

The wild arc Z of $[1; 1.3]$, whose complement $S^3 - Z$ is not an open 3-cell, is the union of the arc Y and its mirror image in the plane $x = 0$. Assume $B = S^3 - (Z \cup \infty)$, where ∞ denotes the point "at infinity". Obviously, the sets A and B are not homeomorphic.

Now we prove topological equivalence of sets $A \times I$ and $B \times I$. Let us translate Y rigidly in the negative direction of x-axis by 4 units. This operation provides us with an isometric copy Y' of Y, which can be separated from Y by a Euclidean 2-sphere. Let Q be a solid sphere $x^2 + y^2 + z^2 \leqslant \varepsilon^2$ whose radius ε is so small that $Z \cap Q$ is a tame arc. Let Q_∞ be the 3-cell $x^2 + y^2 + z^2 \geqslant 16$. In analogy with the definition of the mapping g mentioned before, we can construct such a mapping g' of S^3 on itself that $g'|_{S^3 - (Y \cup Q \cup Y' \cup Q_\infty)}$ is a homeomorphism and the set

$g'(Y \cup Q \cup Y' \cup Q_\infty)$ consists of two different points. So the set $A' = S^3 - (Y \cup Q \cup Y' \cup Q_\infty)$, shown in Fig. 3, is homeomorphic with A.

The set $B' = S^3 - (Z \cup Q \cup Q_\infty)$, shown in Fig. 4, is homeomorphic with B. Now fix $I = (-1, 1)$. It is easy to see that there is such an isotopy h_t, $0 \leqslant t \leqslant 1$, of $S^3 \times [-3, 3]$ on itself that:

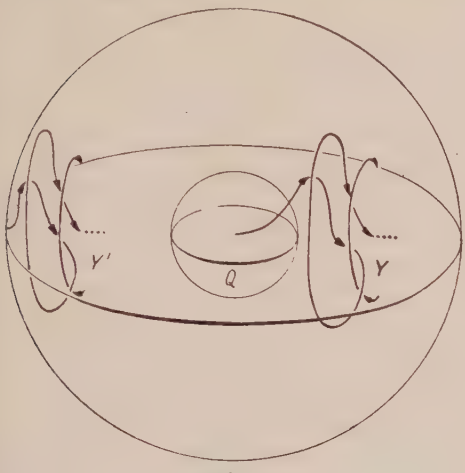

Fig. 3

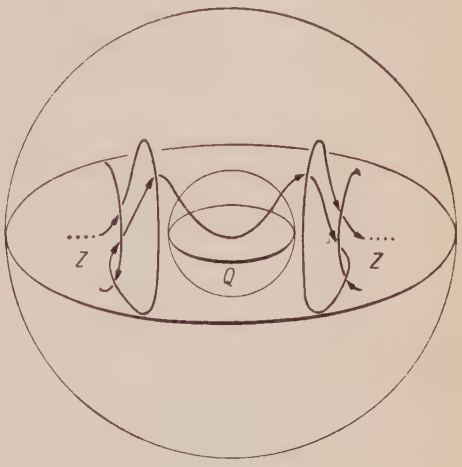

Fig. 4

i) for each t, the map h_t is a homeomorphism of $S^3 \times [-3, 3]$ on istelf;

ii) for each t, h_t is the identity mapping outside of $Q' \times (-2, 2)$, where Q' is the solid half-sphere $x^2 + y^2 + z^2 < 5^2$, $x < 0$;

iii) h_1 carries the set $A' \times I$ on the set $B' \times I$.

This completes the proof.

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MATHEMATICS

On the Theory of Operator-Systems. II.

by

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Presented by A. MOSTOWSKI on April 21, 1958

Introduction

The theory of operator-systems, as annouced in [1], can be extended to more general objects named map-systems. There are many important examples of map-systems which are not operator-systems, and it is very natural to extend the theorems proved in [1] to this case. This is the subject-matter of the present note. Only algebraic theorems concerning map-systems are considered. These results, as well as those given in [1], will be published soon in more extensive form with a number of examples in Fundamenta Mathematicae.

1. The notion of map-system

A map-system is an ordered pair $\mathfrak{A} = (\boldsymbol{S}, \boldsymbol{X})$ where:

a. $\boldsymbol{S}$ is a semigroup written multiplicatively with the unit element I (the cancellation law is not assumed);

b. $\boldsymbol{X}$ is a commutative group written additively; (in what follows $\boldsymbol{S}$ will often be noted $l\mathfrak{A}$ — left member of $\mathfrak{A}$, and similarily $\boldsymbol{X}$ will be denoted $r\mathfrak{A}$ — right member of $\mathfrak{A}$);

c. with each element $A \in \boldsymbol{S}$ there is associated a subgroup $\boldsymbol{G}_A$ of $\boldsymbol{X}$ and with each pair $A \in \boldsymbol{S}$ and $x \in \boldsymbol{X}$ there is associated an element $y = Ax$ (the composition of A and x) in such a way that

1. the mapping $y = Ax$ is a homomorphism of $\boldsymbol{G}_A$ into the group $\boldsymbol{X}$,

2. for every A, B

$$\boldsymbol{G}_{AB} \supset \boldsymbol{G}_B, \qquad B(\boldsymbol{G}_{AB}) \subset \boldsymbol{G}_A$$

and $(AB)x = A(Bx)$ for $x \in G_{AB}$.

If $\boldsymbol{G}_A = \boldsymbol{X}$ for each $A \in \boldsymbol{S}$ then $\mathfrak{A}$ is said to be algebraically closed; $\mathfrak{A}$ is said to be a linear map-system if, and only if, $\boldsymbol{X}$ is a linear space and the condition $x \in \boldsymbol{G}_A$ implies $\lambda x \in \boldsymbol{G}_A$ and $A(\lambda x) = \lambda(Ax)$ for every scalar λ.

$\mathfrak{A}$ is said to be an *operator-system* [1] if, and only if, $\boldsymbol{S}$ is commutative and all the homomorphisms $A \in \boldsymbol{S}$ map their domains $\boldsymbol{G}_A$ onto whole $\boldsymbol{X}$.

Let $\mathfrak{A} = (\boldsymbol{S}, \boldsymbol{X})$ and $\mathfrak{A}' = (\boldsymbol{S}', \boldsymbol{X}')$ be two map-systems (linear map-systems). A pair $\mathfrak{H} = (\boldsymbol{H}, \mathfrak{h})$ is said to be a *homomorphism* of $\mathfrak{A}$ into $\mathfrak{A}'$ [1] if, and only if, $\boldsymbol{H}$ is a homomorphism of $\boldsymbol{S}$ into $\boldsymbol{S}'$, $\boldsymbol{h}$ — a homomorphism (a linear mapping) of $\boldsymbol{X}$ into $\boldsymbol{X}'$, and

$$\boldsymbol{h}(Ax) = (\boldsymbol{H}A)(\boldsymbol{h}x) \quad \text{for} \quad A \in \boldsymbol{S} \quad \text{and} \quad x \in \boldsymbol{G}_A .$$

$\mathfrak{H}$ is named *isomorphism* if, and only if, $\boldsymbol{H}$ and $\boldsymbol{h}$ are one-to-one.

We say that $\mathfrak{H}$ maps $\mathfrak{A}$ *almost onto* $\mathfrak{A}'$ or that $\mathfrak{A}'$ is an *almost image* of $\mathfrak{A}$ if, and only if:

α. $\boldsymbol{H}(l\mathfrak{A}) = l\mathfrak{A}'$

β. for each $x' \in \boldsymbol{X}'$ there are $A_{i,j} \in \boldsymbol{S}$, and $x_j \in \boldsymbol{X}$ $i = 1, 2, \ldots, k_j$, $j = 1, 2, \ldots, n$ such that

$$x' = \sum_{j=1}^{n} \boldsymbol{H}A_{1,j}\left(\boldsymbol{H}A_{2,j} \ldots \left(\boldsymbol{H}A_{k_j,j}(\boldsymbol{h}x_j)\right) \ldots\right) .$$

If $\mathfrak{A}$ is an operator-system and $\mathfrak{A}'$ is algebraically closed, conditions (β) can be replaced by more simple ones:

β′. for each $x' \in \boldsymbol{X}'$ there are $Y \in \boldsymbol{S}$ and $x \in \boldsymbol{X}$ such that

$$x' = (\boldsymbol{H}A)(\boldsymbol{h}x) .$$

Clearly, β implies (β′).

If $\boldsymbol{S}$ is a subsemigroup of $\boldsymbol{S}'$ and $\boldsymbol{H}A = A$ for each $A \in \boldsymbol{S}$ then the homomorphism $\mathfrak{H}$ will be named a *simple homomorphism* and will be denoted by $(\boldsymbol{h})$.

If homomorphisms $\mathfrak{H} = (\boldsymbol{H}, \boldsymbol{h})$, $\mathfrak{H}_1 = (\boldsymbol{H}_1, \boldsymbol{h}_1)$ and $\mathfrak{H}_2 = (\boldsymbol{H}_2, \boldsymbol{h}_2)$ map $\mathfrak{A}_2$ into $\mathfrak{A}$, $\mathfrak{A}_1$ into $\mathfrak{A}$ and $\mathfrak{A}_2$ into $\mathfrak{A}_1$, respectively, then we write $\mathfrak{H} = \mathfrak{H}_1\mathfrak{H}_2$ if, and only if, $\boldsymbol{H} = \boldsymbol{H}_1\boldsymbol{H}_2$ and $\boldsymbol{h} = \boldsymbol{h}_1\boldsymbol{h}_2$.

Let $\mathfrak{A}$ be a map-system (a linear map-system). A function $\mathfrak{J}$ defined on $l\mathfrak{A}$ with values in the set of all subgroups (linear subspaces) of $r\mathfrak{A}$ is said to be an *ideal* of $\mathfrak{A}$ if, and only if,

a. $\mathfrak{J}(AB) \supset \mathfrak{J}(B)$ for $A, B \in l\mathfrak{A}$.

b. For any two decompositions $\Delta_1, \Delta_2, \ldots, \Delta_p$ and $\Delta_1, \Delta_2, \ldots, \Delta_q$ of the set of numbers $1, 2, \ldots, n$ into disjoint subsets and for any elements $A_i, B_i \in l\mathfrak{A}$ $i = 1, 2, \ldots, n$; where $A_i = A_j$ for $i, j \in \Delta_r$ and $A_iB_i = A_jB_j = C_s$ for $i, j \in \Delta_s$ we have

α. $\sum_{i \in \Delta_r} B_i x_i = 0$, $\sum_{i \in \Delta_s} x_i \in \mathfrak{J}(C_s)$ for $r = 1, 2, \ldots, p$ and all $s \neq s_0$

implies

β. $\sum_{i \in \Delta_{s_0}} x_i \in \mathfrak{J}(C_{s_0})$.

THEOREM 1. *If $\mathfrak{J}$ is an ideal of a map-system $\mathfrak{A}$, then $\mathfrak{J}$ satisfies the condition*

c. *$Bx \in J(A)$ is equivalent to $x \in \mathfrak{J}(AB)$ for $A, B \in l\mathfrak{A}$ and $x \in \boldsymbol{G}_B$.*

THEOREM 2. *If a map-system $\mathfrak{A}$ is an operator-system and if the function $\mathfrak{J}$, that maps $l\mathfrak{A}$ into the set of all subgroups of $r\mathfrak{A}$ satisfies conditions* a, *and* c, *then $\mathfrak{J}$ is an ideal of $\mathfrak{A}$.*

COROLLARY 1. *In the case when $\mathfrak{A}$ is an operator-system, the definition of ideal from* [1] *coincides with that given above.*

If $\{\mathfrak{J}_\xi\}$ is a family of ideals of a map-system $\mathfrak{A}$, then the function

$$(\bigcap_\xi \mathfrak{J}_\xi)(A) \overset{\text{df}}{=} \bigcap_\xi \mathfrak{J}_\xi(A) \qquad A \in l\mathfrak{A}$$

is again an ideal and is named the *intersection* of ideals $\mathfrak{J}_\xi$. Hence, in every map-system $\mathfrak{A}$ there is exactly one minimal ideal, i. e. the ideal which is contained in any other ideal of $\mathfrak{A}$ ($\mathfrak{J}$ is contained in $\mathfrak{J}'$ if, and only if, $\mathfrak{J}(A) \subset \mathfrak{J}'(A)$ for each $A \in l\mathfrak{A}$).

Now, we present the generalization of some results of [1] Theorem 3. Let $\mathfrak{A}$ and $\mathfrak{A}'$ be an arbitrary map-system (a linear map-system) and let $\mathfrak{H} = (\boldsymbol{H}, \boldsymbol{h})$ be a homomorphism of $\mathfrak{A}$ into $\mathfrak{A}'$. If $\mathfrak{J}'$ is an ideal of $\mathfrak{A}'$ then the function

$$(\mathfrak{H}^{-1}\mathfrak{J}')(A) \overset{\text{df}}{=} \{x : \boldsymbol{h}x \in \mathfrak{J}'(\boldsymbol{H}A)\}$$

is an ideal of $\mathfrak{A}$.

And here is the generalization of the Fundamental Theorem of [1]:

2. The Fundamental Theorem

I. *Let $\mathfrak{A}$ be an arbitrary map-system (a linear map-system) and let $\mathfrak{J}$ be an ideal of $\mathfrak{A}$. Then there is an algebraically closed map-system (linear map-system) $\mathfrak{A}/\mathfrak{J}$, named the quotient map-system of $\mathfrak{A}$ and $\mathfrak{J}$ and a simple homomorphism (h) that maps A almost onto $\mathfrak{A}/\mathfrak{J}$ in such a way that*

$$(h)^{-1}\mathfrak{J}^M = \mathfrak{J},$$

where $\mathfrak{J}^M$ is the minimal ideal of $\mathfrak{A}/\mathfrak{J}$.

II. *Let $\mathfrak{A}'$ and $\mathfrak{A}''$ be two map-systems (linear map-systems), $\mathfrak{J}'$ and $\mathfrak{J}''$ two ideals of $\mathfrak{A}'$ and $\mathfrak{A}''$, respectively, and $\mathfrak{H} = (\boldsymbol{H}, \boldsymbol{h})$ a homomorphism of $\mathfrak{A}'$ into $\mathfrak{A}''$.*

The condition

$$\mathfrak{H}^{-1}\mathfrak{J} \supset \mathfrak{J}'$$

is true if, and only if, there is exactly one homomorphism $\mathfrak{H}^*$ of $\mathfrak{A}/\mathfrak{J}$ into $\mathfrak{A}''/\mathfrak{J}''$ satisfying the condition

$$(\boldsymbol{h}'')\mathfrak{H} = \mathfrak{H}^*(h').$$

If $\mathfrak{H}$ maps $\mathfrak{A}'$ almost onto $\mathfrak{A}$ then the homomorphism $\mathfrak{H}^*$ maps $\mathfrak{A}'/\mathfrak{I}'$ onto $\mathfrak{A}''/\mathfrak{I}''$. $\mathfrak{H}^*$ is an isomorphism if, and only if, $\mathfrak{H}^{-1}\mathfrak{I}'' = \mathfrak{I}'$; $\mathfrak{H}^*$ is simple if, and only if, $\mathfrak{H}$ is simple.

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Beta-Decays and Time Reversal

by

J. WRZECIONKO

Presented by L. INFELD on April 4, 1958

Introduction

Recently many papers have appeared in which the question of invariance of weak interactions under Wigner time reversal is being investigated. The problem of finding suitable experimental tests for time reversal invariance is much more complicated than that for space inversion invariance.

In this second case an experimental detection of space-pseudoscalar term in the probability function was an unequivocal argument against parity conservation. An experimental detection of a time-pseudoscalar term in the transition probability is, however, not sufficient to conclude that the time reversal invariance is violated.

The results hitherto published [1], [2], [3], [6] can be shortly formulated as follows: if the Coulomb interaction of the β-decay electron with the Coulomb field of the nucleus is neglected, then no time-pseudoscalar terms can appear in the transition probability when the theory is invariant under Wigner's time reversal operation. When the Coulomb effects are taken into account, the coefficients at time pseudoscalars contain some additional terms proportional to $\frac{Ze^2}{p}$ which do not vanish even when time reversal invariance is valid. This is the very reason why it is a difficult question to decide whether the weak interactions are invariant under time reversal operation.

In the present paper, the question of the invariance of weak interaction was reinvestigated without any application of perturbational methods and other simplifying assumptions made by other authors.

The investigations were carried out on the basis of J. Werle's [4] paper, which dealt with the problem of general conditions to be imposed on the transition matrix by $\hat{P}$, $\hat{C}$ or $\hat{T}$ invariance requirements.

We shall here investigate the general form of the transition probability for $1\to 0$-transitions.

S-matrix for Beta decay

We are using here the following representation

$$\vec{\gamma}^{+}=\vec{\gamma};\qquad \gamma_0^{+}=-\gamma_0;\qquad \gamma_5^{+}=\gamma_5;\qquad \gamma_\mu p_\mu=\vec{\gamma}\vec{p}+\gamma_0 p_0\,.$$

Let us discuss the process when a nucleus with spin 1 and momentum $\vec{P}'$ decays into a nucleus with spin 0 and momentum $\vec{P}''$, emitting an electron with momentum $\vec{p}$ and a neutrino with momentum $\vec{q}$. The $\hat{S}$-matrix will have the form

$$\hat{S}=1+i\hat{M}+i\hat{R}\,. \tag{1}$$

$\hat{M}$ refers to the "part" of the matrix describing the process with which we are dealing. Other "parts", irrelevant to the said process, are symbolized by $\hat{R}$. The most general form of $\hat{M}$ for processes describing all possible processes involving two fermion fields and two boson fields, has been given by J. Werle [4]. In our case $\hat{M}$ has the following form:

$$\hat{M}=\int\delta_4(P'-P''-p+q)\prod_{i=1}^{4}\delta(p_i^2+m_i^2)\,d_4p_i\times \tag{2}$$
$$\times\{\Phi_\mu(P')\varphi^{+}(P'')\bar{\psi}_e(p)\Gamma_\mu\psi_\nu(q)+\Phi_\mu^{+}(P')\varphi(P'')\bar{\psi}_\nu(q)\Gamma'_\mu\psi_e(p)\}\,.$$

Φ_μ and φ are field operators representing the nucleus proper in its initial and in its final states, the p_i denote P', P'', p, q — the four-momenta of the particles; m_i — their masses; ψ_e and ψ_ν operators of the electron and neutrino fields.

$$\Gamma_\mu=\sum_{j=1}^{3}\{a_j+\gamma_5 a'_j+(b_j+\gamma_5 b'_j)i\gamma_a P'_a\}N_\mu^j$$
$$\Gamma''_\mu=\sum_{j=1}^{3}N_\mu^j\{c_j-\gamma_5 c'_j+i\gamma_a P'_a(d_j-\gamma_5 d'_j)\}\quad N_\mu^1=p_\mu;\quad N_\mu^2=q_\mu;\quad N_\mu^3=i\gamma_\mu \tag{3}$$

a_j, b_j, c_j, d_j are unknown scalar functions.

The field operators are expanded according to states of definite momenta and polarizations

$$\Phi_\mu=\sum_{\lambda,k}C_\lambda^1(k)\varkappa^1(k)e_\mu^\lambda+\text{c.c}\,,$$
$$\varphi=\sum_{k}C^0(k)\varkappa^0(k)+\text{c.c}\,, \tag{4}$$
$$\psi=\sum_{k,s}C_s^{1/2}(k)U_s(k)+\text{ch.c}\,.$$

$C^1_\lambda(k)$ are operators of the annihilation of bosons with spin 1, four-momentum k and polarization λ. $C^0(k)$ are operators of the annihilation of bosons with spin 0.

$C^{1/2}_s(k)$ are operators of the annihilation of fermions with spin $\frac{1}{2}$, four-momentum k and polarization s.

$\varkappa^1$, $\varkappa^0$ are known functions depending on the normalization.

$U_s(k)$ is a spinor describing an electron with four-momentum k and polarization s.

The probability of the process is

$$W \sim |\langle f|\hat{S}|i\rangle|^2 = |\langle f|\hat{M}|i\rangle|^2$$

with

$$\begin{aligned} |i\rangle &= C^{1+}_\lambda(P')|0\rangle\,, \\ |f\rangle &= C^{0+}(P'')\,C^{1/2+}_{s_e}(p)\,C^{1/2+}_{s_\nu}(q)|0\rangle\,. \end{aligned} \tag{5}$$

After summing over the neutrino spin we arrive at the formula:

$$\begin{aligned} W \sim Sp\,\{[(\hat{\vec{a}}\vec{q})+|\vec{q}|](\Gamma_\mu e^\lambda_\mu)^+\gamma_0^+[1+\vec{\sigma}\vec{\xi}+\beta\vec{\sigma}\vec{\eta}]\times \\ \times[(\hat{\vec{a}}\vec{p})+\beta m+E_p]\gamma_0(\Gamma_\nu e^\lambda_\nu)\}\,, \end{aligned} \tag{6}$$

where $\vec{\xi}$ and $\vec{\eta}$ are the vectors of the longitudinal and transversal electron polarization.

Beta decay from oriented nuclei

We are dealing here with oriented nuclei. Passing over to a system in which the initial nucleus is at rest and assuming that $\vec{e} = \vec{e}_1 + i\vec{e}_2$ (i. e. projection of the nuclear spin on $\vec{e}_3$ is $+1$) we obtain

$$\begin{aligned} Sp\,\{-\,,,\,-\} &= [\text{time scalar}] + \\ &+\vec{I}(\vec{p}\times\vec{q})[\varkappa^0_1+\varkappa^1_1(\vec{p}\vec{q})+\varkappa^2_1(\vec{I}\vec{p})+\varkappa^3_1(\vec{I}\vec{q})+ \\ &\qquad +\varkappa^4_1(\vec{\xi}\vec{p})+\varkappa^5_1(\vec{\xi}\vec{q})+\varkappa^6_1(\vec{\eta}\vec{q})]+ \\ &+\vec{\eta}(\vec{p}\times\vec{q})[\varkappa^0_2+\varkappa^1_2[\vec{p}^2-(\vec{I}\vec{p})^2]+\varkappa^2_2[\vec{q}^2-(\vec{I}\vec{q})^2]+ \\ &\qquad +\varkappa^3_2[(\vec{p}\vec{q})-(\vec{I}\vec{p})(\vec{I}\vec{q})]]+ \\ &+\vec{I}(\vec{\eta}\times\vec{p})[\varkappa^0_3+\varkappa^1_3(\vec{p}\vec{q})+\varkappa^2_3(\vec{I}\vec{p})+\varkappa^3_3(\vec{I}\vec{q})]+ \\ &+\vec{I}(\vec{q}\times\vec{\eta})[\varkappa^0_4+\varkappa^1_4(\vec{p}\vec{q})+\varkappa^2_4(\vec{I}\vec{p})+\varkappa^3_4(\vec{I}\vec{q})]+ \\ &+\vec{I}(\vec{q}\times\vec{\xi})[\varkappa^0_5+\varkappa^1_5(\vec{I}\vec{p})+\varkappa^2_5(\vec{I}\vec{q})]\,. \end{aligned} \tag{7}$$

Owing to lack of space we have written only that part of the probability of the transition that contains so-called time pseudoscalars.

The factors $\varkappa^j_i$ are given in the Appendix. $\vec{I}$ is the nuclear spin vector.

Eq. (8) contains five time pseudoscalars, viz:

$$\vec{I}(\vec{p}\times\vec{q}); \quad \vec{\eta}(\vec{p}\times\vec{q}); \quad \vec{I}(\vec{\eta}\times\vec{p}); \quad \vec{I}(\vec{q}\times\vec{\eta}); \quad \vec{I}(\vec{q}\times\vec{\xi}). \tag{8}$$

It can be seen from the Appendix that the coefficients at these five time pseudoscalars do not vanish in general even for a theory which is invariant under time reversal. In fact, from condition [4]

$$\hat{T}\hat{M}\hat{T}^{-1} = \hat{M}^{+} \tag{9}$$

we find that

$$\begin{aligned} &\eta_T a_k = c_k; \quad \eta_T b_k = d_k \\ &\eta_T a'_k = c'_k; \quad \eta_T b'_k = d'_k \end{aligned} \qquad k = 1, 2, 3. \tag{10}$$

However, if we add to condition (9), the hermiticity condition

$$\hat{M} = \hat{M}^{+} \tag{11}$$

we obtain

$$\begin{aligned} &\eta_T a_k = a_k^*; \quad \eta_T b_k = b_k^* \\ &\eta_T a'_k = a_k'^*; \quad \eta_T b'_k = b_k'^* \end{aligned} \qquad k = 1, 2, 3.$$

(11) is just sufficient to make all the factors at time pseudoscalars equal to zero.

Hermiticity condition for the transition matrix is usually justified in the case of those slow decays in which the interactions in the final state can be neglected.

Concluding remarks

The above fact is an expression of a more general law. For, let us consider any process described by a transition matrix $\hat{M}$. If the states $|i\rangle$ and $|f\rangle$ are free particle states and if $\hat{M}$ is a hermitian operator ($\hat{M} = \hat{M}^{+}$), then the condition of invariance with respect to a Wigner time reversal

$$\hat{T}\hat{M}\hat{T}^{-1} = \hat{M}^{+} \tag{12}$$

excludes the possibility of the occurrence in the probability of the given process of terms proportional to the time pseudoscalars constructed from observables.

On the basis of the above law we shall analyse the transition probability for a reaction between two bosons and two fermions. For the sake of convenience we write the operator $\hat{M}$ given in [4] in the form

$$M = \int \delta_4(P' - P'' - p + q) \prod_{i=1}^{4} \delta(p_i^2 + m_i^2) \times [\Phi_r(P')\Phi_s^{+}(P'')\bar{\psi}_1(p)\Gamma_{rs}\psi_2(q) + \\ + \Phi_r^{+}(P')\Phi_s(P'')\bar{\psi}_2(q)\Gamma'_{rs}\psi(p)], \tag{13}$$

where p_i and m_i are the 4-momenta and masses of the particles taking part in the process.

$$\Gamma_{rs} = \sum_{k=1}^{N} A_k N_{rs}^k + \sum_{k=1}^{N} A'_k \gamma_5 N_{rs}^k$$

$$\Gamma'_{rs} = \sum_{k=1}^{N} B_k \gamma_4 N_{rs}^{k+} \gamma_4 - \sum_{k=1}^{N} B'_k \gamma_4 N_{rs}^{k+} \gamma_4 \gamma_5 .$$

A_k, A'_k, B_k, B'_k are unknown scalar functions. The N_{rs}^k denote N independent covariant quantities constructed from the 4-momenta of the particles taking part in the process, and from the Dirac γ_μ-matrix, which, together with the spinor functions ψ_1 and ψ_2, form tensors whose indices are contracted with the tensor indices of the Boson fields Φ_r and Φ_s. The transition probability for processes of this type can be written in the form

$$W = |\langle f|\hat{S}|i\rangle|^2 = |\langle f|\hat{M}|i\rangle|^2 \sim \sum_{n=1}^{N} |A_n|^2 |\lambda_n|_1^2 + \sum_{n=1}^{N} |A'_n|^2 |\lambda'_n|^2 +$$
$$+ 2 \sum_{k>m}^{N} \{\mathrm{Re}[A_k A_m^*]\,\mathrm{Re}[\lambda^k \lambda^{m*}] + \mathrm{Im}\, A_k A_m^*]\,\mathrm{Im}[\lambda^k \lambda^{m*}] +$$
$$+ \mathrm{Re}[A'_k A'^*_m]\,\mathrm{Re}[\lambda^{k'} \lambda^{m'*}] + \mathrm{Im}[A'_k A'^*_m]\,\mathrm{Im}[\lambda^{k'} \lambda^{m'*}]\}$$
$$+ 2 \sum_{k,m}^{N} \{\mathrm{Re}[A_k A'^*_m]\,\mathrm{Re}[\lambda^k \lambda^{m'*}] + \mathrm{Im}[A_k A'^*_m]\,\mathrm{Im}[\lambda^k \lambda^{m'*}]\},$$

where

$$\lambda^k = \langle f|\Phi_r \Phi_s^+ \bar{\psi}_1 N_{rs}^k \psi_2|i\rangle ,$$
$$\lambda^{k'} = \langle f|\Phi_r \Phi_s^+ \bar{\psi}_1 \gamma_5 N_{rs}^k \psi_2|i\rangle .$$

$|i\rangle$ and $|f\rangle$ are free particle states ($\gamma_5 = \gamma_5^+$).

All expressions of the form

$$\Phi_r \Phi_s^+ \bar{\psi}_1 N_{rs}^k \psi_2 ; \qquad \Phi_r \Phi_s^+ \bar{\psi}_1 \gamma_5 N_{rs}^k \psi_2$$

are transformed in the same way under the Wigner time reversal. On the other hand,

(14)
$$\begin{aligned}
\mathrm{Re}[A_k A_m^*] &\to \mathrm{Re}[A_k A_m^*] \\
\mathrm{Re}[A'_k A'^*_m] &\to \mathrm{Re}[A'_k A'^*_m] \\
\mathrm{Re}[A_k A'^*_m] &\to \mathrm{Re}[A_k A'^*_m] \\
\mathrm{Im}[A_k A_m^*] &\to -\mathrm{Im}[A_k A_m^*] \\
\mathrm{Im}[A'_k A'^*_m] &\to -\mathrm{Im}[A'_k A'^*_m] \\
\mathrm{Im}[A_k A'^*_m] &\to -\mathrm{Im}[A_k A'^*_m]
\end{aligned} \qquad k, m = 1 \ldots N .$$

The other side of condition (12) gives

$$\begin{aligned} \eta_T A_k^* &= B_k^* \\ \eta_T A_k'^* &= B_k'^* \end{aligned} \qquad k = 1 \ldots N$$

while the condition $\hat{M} = \hat{M}^+$ gives $A_k^* = B_k$; $A_k'^* = B_k'$.

If both these conditions are satisfied for $\hat{M}$ simultaneously, then all the term $\mathrm{Im}[A_k A_m^*]$; $\mathrm{Im}[A_k' A_m'^*]$ $\mathrm{Im}[A_k A_m'^*]$ vanish identically.

From the other side it can be easily shown that, if $\hat{M}$ is a hermitian operator and conditions (12) are fulfilled, then the transition probability is an even function of momenta $\vec{p}_i$ and polarization vectors $\vec{s}_i$

$$W(\ldots \vec{p}_i \ldots; \ldots \vec{s}_i \ldots) = W(\ldots -\vec{p}_i \ldots; \ldots -\vec{s}_i \ldots).$$

Thus the expressions $\mathrm{Re}[\lambda^k \lambda^{m*}]$, $\mathrm{Re}[\lambda^{k'} \lambda^{m'*}]$, $\mathrm{Re}[\lambda^k \lambda^{m'*}]$ cannot contain any time pseudoscalars built from the vectors $\vec{p}_i$ and $\vec{s}_i$. These pseudoscalars may enter only the expressions $\mathrm{Im}[\lambda^k \lambda^{m*}]$, $\mathrm{Im}[\lambda^{k'} \lambda^{m'*}]$, $\mathrm{Im}[\lambda^k \lambda^{m'*}]$.

Thus we see that the invariance under time reversal and hermiticity condition make all the time-pseudoscalars vanish. However, it should be noted that the neglect of all final state interactions is, in general, not sufficient to justify the hermitian character of the transition matrix.

The author wishes to thank Professor Werle for suggesting the problem and for many valuable discussions.

Appendix

Symbols: E_p — electron energy, M — nucleus mass in the initial state, m — electron mass.

$$\begin{aligned}
\varkappa_1^0 &= 2\,\mathrm{Im}[ME_p(a_1^* b_3 + a_1'^* b_3' - b_1^* a_3 - b_1'^* a_3') - M|\vec{q}\,|(b_2^* a_3 + b_2'^* a_3' + a_2^* b_3 + a_2'^* b_3') + \\
&\quad + E_p|\vec{q}\,|(M^2 b_1 b_2^* + M^2 b_1' b_2'^* + a_1 a_2^* + a_2' a_2'^*)] + 2\,\mathrm{Im}[m(a_1^* a_3 - a_1'^* a_2' + \\
&\quad + M^2 b_1'^* b_3' - M^2 b_1^* b_3) + 2mM|\vec{q}\,|(b_1' a_2'^* - b_1 a_2^* + a_1^* b_2 - a_1'^* b_2')] \\
\varkappa_1^1 &= 2\,\mathrm{Im}[M^2(b_1 b_2^* + b_1' b_2'^*) - (a_1 a_2^* + a_1' a_2'^*)] \\
\varkappa_1^2 &= -2M\,\mathrm{Im}[b_1' a_3^* + b_1 a_3'^* + a_1^* b_3' + a_1'^* b_3] \\
\varkappa_1^3 &= -2M\,\mathrm{Im}[b_2' a_3^* + b_2 a_3'^* + a_2^* b_3' + a_2'^* b_3] \\
\varkappa_1^4 &= 2\,\mathrm{Im}[mM(a_1 b_3'^* - a_1'^* b_3 + b_1'^* a_3 - b_1 a_3'^*) + |\vec{q}\,|(a_1 a_2'^* - a_1'^* a_2 + \\
&\quad + M^2 b_1 b_2'^* - M^2 b_1'^* b_2)] \\
\varkappa_1^5 &= 2mM\,\mathrm{Im}[a_2'^* b_1 - a_2^* b_1' + a_1^* b_2' - a_1'^* b_2] + 2E_p\,\mathrm{Im}[M^2(b_1 b_2'^* - b_2 b_1'^*) + \\
&\quad a_2 a_1'^* - a_2'^* a_1] \\
\varkappa_1^6 &= 2\,\mathrm{Im}[M^2(b_2 b_3'^* - b_2' b_3^*) + a_2' a_3^* - a_2 a_3'^* + E_p M(a_2'^* b_1 - a_2^* b_1' + \\
&\quad a_1'^* b_2 - a_1^* b_2')] + 2m\,\mathrm{Im}[M^2(b_1 b_2'^* - b_2 b_1'^*) + a_2 a_1'^* - a_1 a_2'^*] \\
\varkappa_2^0 &= 2\,\mathrm{Im}[M^2(|\vec{q}\,|b_2'^* b_3' - |\vec{q}\,|b_2^* b_3 + E_p b_1^* b_3 - E_p b_1^* b_3') + M(a_3' b_3'^* - a_3 b_3^*) + \\
&\quad + |\vec{q}\,|(a_3' a_2'^* - a_3 a_2^*) + E_p(a_3 a_1^* - a_3' a_1'^*)] + 2Mm\,\mathrm{Im}[b_1'^* a_3' + b_1^* a_3 - a_1'^* b_3' - a_1^* b_3]
\end{aligned}$$

$$\varkappa_2^1 = M\,\mathrm{Im}[b_1 a_2^* - b_1' a_1'^*]$$

$$\varkappa_2^2 = M\,\mathrm{Im}[b_2 a_2^* - b_2' a_2'^*]$$

$$\varkappa_2^3 = M\,\mathrm{Im}[b_2 a_1^* - b_2' a_1'^* + b_1 a_2^* - b_1' a_2'^*]$$

$$\varkappa_3^0 = 2E_p|\vec{q}\,|\,\mathrm{Im}[M^2(b_1^* b_3' - b_1'^* b_3) + (a_1^* a_3' - a_1'^* a_3)] + \\ + 2Mm\,|\vec{q}\,|\,\mathrm{Im}[a_1' b_3^* - a_1^* b_3' - b_1'^* a_3 + b_1 a_3'^*]$$

$$\varkappa_3^1 = 2\,\mathrm{Im}[(a_1'^* a_3 - a_1^* a_3') + M^2(b_1^* b_3' - b_1'^* b_3)]$$

$$\varkappa_3^2 = 2\,|\vec{q}\,|\,\mathrm{Im}[M^2(b_1^* b_3 - b_1'^* b_3') + (a_1'^* a_3' - a_1^* a_3)]$$

$$\varkappa_3^3 = 2\,\mathrm{Im}[M^2|\vec{q}\,|(b_2^* b_3 - b_2'^* b_3') + |\vec{q}\,|(a_2^* a_3 - a_2'^* a_3') + 2M(a_3' b_3'^* - a_3 b_3^*)]$$

$$\varkappa_4^0 = 2E_p|\vec{q}\,|\,\mathrm{Im}[M^2(b_2 b_3'^* - b_2' b_3^*) + (a_2 a_3'^* - a_2' a_3^*)] + \\ + 2m\,|\vec{q}\,|M\,\mathrm{Im}[b_2'^* a_3 + a_2^* b_3' - b_2 a_3'^* - a_2' b_3^*]$$

$$\varkappa_4^1 = 2\,\mathrm{Im}[M^2(b_2 b_3'^* - b_2' b_3^*) + a_2 a_3'^* - a_2' a_3^*]$$

$$\varkappa_4^2 = 2mM\,\mathrm{Im}[a_1^* b_3 + a_1'^* b_3' - b_1^* a_3 - b_1'^* a_3'] - \\ - 2E_p\,\mathrm{Im}[M^2(b_1^* b_3 - b_1'^* b_3') + a_1^* a_3 - a_1'^* a_3']$$

$$\varkappa_4^3 = 2mM\,\mathrm{Im}[a_2^* b_3 + a_2'^* b_3' - b_2^* a_3 - b_2'^* a_3'] - \\ - 2E_p\,\mathrm{Im}[M^2(b_2^* b_3 - b_2'^* b_3') + (a_2^* a_3 - a_2'^* a_3')]$$

$$\varkappa_5^0 = 2\,|\vec{q}\,|ME_p\,\mathrm{Im}[b_2'^* a_3 + a_2^* b_3' - a_2' b_3^* - b_2 a_3'^*] + \\ + 2m\,|\vec{q}\,|\,\mathrm{Im}[M^2(b_2'^* b_3 - b_2^* b_3') + (a_2'^* a_3 - a_2^* a_3')]$$

$$\varkappa_5^1 = 2m\,\mathrm{Im}[M^2(b_1'^* b_3' - b_1^* b_3) + a_1'^* a_3' - a_1^* a_3] + \\ + 2mME_p\,\mathrm{Im}[a_1^* b_3 + a_1'^* b_3' - b_1^* a_3 - b_1'^* a_3']$$

$$\varkappa_5^2 = 2m\,\mathrm{Im}[M^2(b_2'^* b_3' - b_2^* b_3) + a_2^* a_3 - a_2'^* a_3'] + \\ + 2ME_p\,\mathrm{Im}[a_2^* b_3 + a_2'^* b_3' - b_2^* a_3 - b_2'^* a_3'].$$

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Tests for Time Reversal Invariance and Feynman – Gell-Mann Theory in Allowed Beta Transitions

by

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Up till now it is not clear whether β-decay interactions are invariant under time reversal. Several possible tests for the validity of time reversal invariance have been given by other authors, but not all consequences have been investigated. In this paper only allowed beta decays of non-oriented nuclei are investigated. There are, in principle, two possible arrangements in which terms that are not invariant under time reversal can be detected.

One test was proposed by M. Morita and R. S. Morita [1]. For pure Gamow-Teller transitions ($\Delta J = \pm 1$) they obtained the electron-neutrino angular correlation function of the form

$$1 + (p/3E) f(g) \cos\vartheta \,.$$

If a dependence of $f(g)$ on Z or on the electron momentum can be observed, then time reversal invariance does not hold in beta decay. Another test was given by Jackson, Treiman and Wyld [2]. They proposed a simultaneous measurement of the electron-neutrino angular correlation and of the electron polarization in the direction perpendicular to the decay plane, which is determined by the momenta of the electron and the neutrino. When Coulomb corrections are neglected, the existence of such a polarization is possible only due to time reversal non-invariance. In further calculations [3] Coulomb corrections to the transversal polarization were obtained, which did not violate time reversal invariance.

In the quoted papers, pseudoscalar couplings have been omitted, although in view of the present situation in beta decay this seems to remain an open question. Also an analysis of the proposed tests from the standpoint of the new theory of Feynman and Gell-Mann [4] should be of interest.

It seemed advisable to perform calculations with correct Coulomb functions. The exact method developed recently by Deloff [5] was applied.

This is a generalization of the method of Cutkosky [6]. A solution of Dirac equation in Coulomb field is represented as the product of a certain operator A and the usual plane-wave spinor u. The operator A has the following form

$$A = F_1 + \beta F_2 + i\vec{\alpha}\cdot\vec{r}(F_3 + \beta F_4)\,, \tag{1}$$

where the F_i are functions of $\vec{p}$ and $\vec{r}$, given by Deloff. In allowed transitions we take into account only the first terms of his expansion of F_1 and F_2.

The interaction Hamiltonian density is assumed to be a sum of the usual five invariants which are taken in the following form:

$$\begin{aligned}
H_S &= \sum C_S(\Psi_p^+ \beta \Psi_n)(\psi_e^+ \beta\varphi_\nu + a_S\psi_e^+\beta\gamma_5\varphi_\nu) + \text{h. c.}\\
H_V &= \sum C_V(\Psi_p^+\Psi_n)(\psi_e^+\varphi_\nu + a_V\psi_e^+\gamma_5\varphi_\nu) + \text{h. c.}\\
H_T &= \sum C_T(\Psi_p^+\beta\vec{\sigma}\Psi_n)(\psi_e^+\beta\vec{\sigma}\varphi_\nu + a_T\psi_e^+\beta\vec{\sigma}\gamma_5\varphi_\nu) + \text{h. c.}\\
H_A &= \sum C_A(\Psi_p^+\vec{\sigma}\Psi_n)(\psi_e^+\vec{\sigma}\varphi_\nu + a_A\psi_e^+\vec{\sigma}\gamma_5\varphi_\nu) + \text{h. c.}\\
H_P &= \sum C_P(\Psi_p^+\beta\gamma_5\Psi_n)(\psi_e^+\beta\gamma_5\varphi_\nu + a_P\psi_e^+\beta\varphi_\nu) + \text{h. c.}
\end{aligned} \tag{2}$$

Though the nucleons are treated nonrelativistically, pseudoscalar couplings are retained, because the coefficient C_P might be greater than the remaining coefficients.

We shall calculate the squared transition matrix element from an initial nucleus state to a final nucleus state with the emission of an electron (positron) and an antineutrino (neutrino). Let us introduce the co-ordinate system in the following way: the electron momentum $\vec{p}$ is directed along the x-axis and the XOY-plane is fixed by the momenta $\vec{p}$ and $\vec{q}$ of the electron and neutrino. For the β-particle in the final state of the system a certain probability is assumed for spin orientation in the direction $\vec{\zeta}$. The three components of the unit vector $\vec{\zeta}$ are denoted in our co-ordinate system as (n, t_y, t_z). The spin states are referred to the rest system of the moving particle.

The usual trace technique is used. We write the electron wave function in the form of a product of the operator A multiplied by an ordinary plane-wave spinor. The final electron (or positron) states are eigenstates of the two operators $\Lambda^\pm$ and λ. The operators $\Lambda^\pm$ are the energy-state projection operators. The spin projection operator λ is taken in the form

$$\tfrac{1}{2}(1 + \vec{\sigma}\cdot\vec{n} + \beta\vec{\sigma}\cdot\vec{t})$$

for electrons and in the form

$$\tfrac{1}{2}(1 - \vec{\sigma}\cdot\vec{n} + \beta\vec{\sigma}\cdot\vec{t})$$

for positrons. The result of the calculations is

$$(3)\quad |\langle H(\vec{p},\vec{q},Z,\vec{\zeta})\rangle|^2 = \frac{1}{8E}F(\pm Z,E)\{E\cdot A - p\cdot\cos(\vec{p},\vec{q})\cdot B \pm \gamma\cdot C \mp$$
$$\mp \alpha Z\cdot\cos(\vec{p},\vec{q})D + n[\pm p\cdot I \mp E\cdot\cos(\vec{p},\vec{q})\cdot J - \gamma\cos(\vec{p},\vec{q})\cdot L + \alpha Z\cdot K] +$$
$$+ t_y\sin(\vec{p},\vec{q})\cdot[\mp\gamma\cdot J - E\cdot L] + t_z\sin(\vec{p},\vec{q})\cdot[\mp \alpha Z\cdot B + p\cdot D]\},$$

where the upper signs refer to electron decay, the lower ones – to positron decay. Energy E and the momenta are in the units of mc^2 and mc respectively, where m denotes the rest mass of the electron, and c – the velocity of light. α is the fine structure constant; Z is the atomic number of the final nucleus; $\gamma = \sqrt{1-\alpha^2Z^2}$. $F(\pm Z, E)$ is the Fermi function. The coefficients $A \ldots L$, which depend on the coupling constants and nuclear matrix elements, are given in the Appendix. The fourth, eighth and eleventh terms in the formula (3) are the Coulomb corrections.

The transversal z-polarization, defined as usual, is given by

$$(4)\quad P_z(\vec{p},\vec{q},Z) = \frac{\sin(\vec{p},\vec{q})\cdot[\mp\alpha Z\cdot B + p\cdot D]}{E\cdot A - p\cdot\cos(\vec{p},\vec{q})\cdot B \pm \gamma\cdot C \mp Z\alpha\cdot D\cdot\cos(\vec{p},\vec{q})}.$$

The terms proportional to αZ are the Coulomb terms. The Coulomb term in the numerator exists even in the case of parity conservation and time reversal invariance. This means that in the decay of heavy nuclei we may have β-particles with a certain degree of transversal polarization which does not violate time reversal invariance. The second term in the numerator vanishes when time reversal invariance is assumed. It follows that the existence of the z-polarization can be a definite test in favour of time reversal non-invariance when the Coulomb corrections are small, i. e. in the decay of the neutron and light nuclei.

This test, however, can be slightly modified for the decay of heavier nuclei. The Z dependence of the z-polarization will reveal the invariance of the beta interaction under time reversal. The starting point is the following: When all other parameters are fixed, the Z dependence of the z-polarization is of a simple form for two kinds of allowed beta transitions:

1) pure Gamow-Teller transitions ($\Delta J = \pm 1$);

2) pure Fermi transitions ($0 \to 0$ transitions).

These two special cases will be investigated in this paper. They are of great interest because in the polarization formulae the nuclear matrix elements in the numerator and denominator cancel and a simple dependence on $\vec{p}$, $\vec{q}$, and Z is evident. Let us introduce two sets of coefficients which depend only on coupling constants. The coefficients $A^{GT}, \ldots, L^{GT}$ are given by the formulae of the Appendix in which we put $M_{GT} = 1$ and $M_F = M_P = 0$. The coefficients $A^F, \ldots, L^F$ are given by the formulae of the Appendix for $M_F = 1$ and $M_{GT} = M_P = 0$. In

this notation, which will be used throughout this paper, the Z dependence of the z-polarization for pure Gamow-Teller transitions has the following form

$$(5)\quad P_z^{GT}(Z) = \frac{\sin(\vec{p},\vec{q})[\mp \alpha Z \cdot B^{GT} + p \cdot D^{GT}]}{E \cdot A^{GT} - p \cdot \cos(\vec{p},\vec{q}) \cdot B^{GT} \pm \gamma \cdot C^{GT} \mp Z \cdot D^{GT} \cdot \cos(\vec{p},\vec{q})}.$$

The form of $P_z(Z)$ for pure Fermi transitions can be obtained from (5) by changing all the coefficients $A^{GT} \ldots L^{GT}$ into $A^F \ldots L^F$. The formula (5) can be used for the following experimental test: The z-polarization has to be measured only for those β-particles from nuclei of different atomic number which have the same energies and which are moving in the same direction with respect to the neutrino. If the experimental data can be fitted only for the nonvanishing constant D^{GT} (or D^F), this will indicate that time reversal invariance is not valid. Under the assumption that the Fierz interference terms vanish, another formulation can be given to this test for time reversal invariance. Namely, if the measurements of $P_z(Z)$ do not indicate a linear dependence on Z (but a dependence like $\frac{a+bZ}{c+dZ}$ with nonvanishing d), this phenomenon is a strong argument against time reversal invariance. A linear dependence of the z-polarization on Z cannot, however, give us any definite answer because the constant D^{GT} (and/or D^F) can be equal to zero for several other reasons, e. g. when the Feynman-Gell-Mann theory is assumed. It should be emphasized that all the experiments with non-oriented nuclei discussed in this paper, can give no answer to the question of time reversal invariance if the Feynman-Gell-Mann theory is valid, because then all the interference terms containing imaginary parts of the products of coupling constants are equal to zero.

While it is not easy to perform such an experiment as the one mentioned above (because several isotopes should be examined), it is a simpler problem to investigate the dependence of P_z on the electron momentum and the electron-neutrino correlation angle for one kind of nuclei. The general form of this dependence is:

$$(6)\quad P_z(\vec{p},\vec{q}) = \frac{(\mp \alpha Z \cdot b_1 + b_2 p) \cdot \sin(\vec{p},\vec{q})}{b_3\sqrt{p^2+1} - b_1 \cdot p \cdot \cos(\vec{p},\vec{q}) \pm b_4 \mp \alpha Z \cdot b_2 \cdot \cos(\vec{p},\vec{q})},$$

where the b_i are constants, which can be easily found from comparison with formula (4). No restriction to pure Fermi or pure Gamow-Teller transitions is now necessary.

If the experimental data can be fitted only for $b_2 \neq 0$, this will be a sufficient argument against time reversal invariance (and also against the theory of Feynman and Gell-Mann). Should the experiment show that the coefficient b_2 is equal to zero, one cannot conclude that time

reversal invariance holds in beta decay. It is, however, possible to decide whether the theory of Feynman and Gell-Mann is valid or not. In this theory the expression (6) has the form

$$P_z^{GT}(\vec{p},\vec{q}) = \frac{\mp aZ \sin(\vec{p},\vec{q})}{3E - p\cdot\cos(\vec{p},\vec{q})} \tag{6a}$$

for all allowed pure GT transitions, and

$$P_z^{F}(\vec{p},\vec{q}) = \frac{\pm aZ\cdot\sin(\vec{p},\vec{q})}{E + p\cdot\cos(\vec{p},\vec{q})} \tag{6b}$$

for allowed pure F transitions.

The experimental demonstration of the validity of (6a) or (6b) could be regarded as a confirmation of the Feynman-Gell-Mann theory.

The test proposed by Morita and Morita [1] for time reversal invariance can be slightly modified to become a test for the Feynman-Gell-Mann theory. If this theory is valid, the electron-neutrino angular correlation function should have the usual simple form:

$E - \frac{1}{3}p\cdot\cos(\vec{p},\vec{q})$ for pure Gamow-Teller transitions, and
$E + p\cdot\cos(\vec{p},\vec{q})$ for pure Fermi transitions.

The demonstration of any other type of dependence would indicate that the universal theory of Fermi interactions of Feynman and Gell-Mann is not valid. The recoil experiment with ^{6}He seems to be in contradiction with this theory and a re-examination is necessary.

Besides the two tests, proposed in [1] and [2] there is also a third one, which may be used as a test either for the time reversal invariance or for the validity of the Feynman-Gell-Mann theory. This is the measurement of the longitudinal component of the polarization of the β-particle as a function of the atomic number, electron momentum and the electron-neutrino correlation angle. The longitudinal component of the polarization of the $\beta^{\mp}$-particle is given by

$$P_x(\vec{p},\vec{q},Z) = \frac{\pm p\cdot I \mp E\cdot\cos(\vec{p},\vec{q})\cdot J - \gamma\cos(\vec{p},\vec{q})\cdot L + aZ\cdot K}{E + A - p\cdot\cos(\vec{p},\vec{q})\cdot B \pm \gamma\cdot C \mp aZ\cdot D\cdot\cos(\vec{p},\vec{q})}. \tag{7}$$

The general form of the Z dependence for pure GT transitions is as follows:

$$P_x^{GT}(Z) = \frac{\pm c_1 - c_2\sqrt{1-a^2Z^2} + d_1\cdot Z}{c_3 \pm c_4\sqrt{1-a^2Z^2} \mp d_2\cdot Z}. \tag{8}$$

A similar expression can be obtained for pure Fermi transitions. The constants c_i and d_i are connected with the coefficients $A^{GT} \ldots L^{GT}$:

$$c_1 = p\cdot I^{GT} - E\cdot\cos(\vec{p},\vec{q})\cdot J^{GT} \qquad c_4 = C^{GT}$$
$$c_2 = \cos(\vec{p},\vec{q})\cdot L^{GT} \qquad d_1 = aK^{GT}$$
$$c_3 = E\cdot A^{GT} - p\cdot\cos(\vec{p},\vec{q})\cdot B^{GT} \qquad d_2 = aD^{GT}\cdot\cos(\vec{p},\vec{q}).$$

If one finds that, to fit the experimental data, at least one of the constants d_i cannot be equal to zero, this would indicate that the invariance under time reversal does not hold.

Also the dependence of the longitudinal component of the polarization on the electron momentum and the electron-neutrino correlation angle can be examined for one kind of nuclei. The general form of this dependence is

$$P_x(\vec{p},\vec{q}) = \frac{e_1\cdot p + \cos(\vec{p},\vec{q})\,(\sqrt{p^2+1}-e_2)e_3 + e_4}{e_5\sqrt{p^2+1} + e_6\cdot p\cdot\cos(\vec{p},\vec{q}) + e_7 + e_8\cdot\cos(\vec{p},\vec{q})} \tag{9}$$

without restriction to pure Gamow-Teller or pure Fermi transitions. The constants e_i can be found easily from comparison with (7).

If it can be deduced from the experiment that the constants e_4 and e_8 cannot be equal to zero, then time reversal invariance will not hold. A sufficient condition for this is to observe a nonvanishing longitudinal polarization of the β-particles, emitted in a direction perpendicular to the neutrino direction, in the limit when their momentum tends to zero.

If the constants e_4 and e_8 must be taken to be equal to zero in order to fit the experimental data, a simple test for the validity of the Feynman Gell-Mann theory can be given. If this theory is valid, then we should have a correlation dependence of the type

$$P_x^{GT}(\vec{p},\vec{q}) = \mp\frac{3p - E\cos(\vec{p},\vec{q})}{3E - p\cdot\cos(\vec{p},\vec{q})} \tag{9a}$$

in pure Gamow-Teller transitions, and

$$P_x^{F}(\vec{p},\vec{q}) = \mp\frac{p + E\cdot\cos(\vec{p},\vec{q})}{E + p\cdot\cos(\vec{p},\vec{q})} \tag{9b}$$

in pure Fermi transitions.

The author wishes to express his indebtedness to Asst. Prof. J. Werle for advice and helpful discussions.

Appendix

In the following formulae M_F is the Fermi nuclear matrix element, M_{GT} is the Gamow-Teller matrix element and M_P is the pseudoscalar matrix element $\int \beta\gamma_5$.

$$A = \left[|C_S|^2(1+|a_S|^2) + |C_V|^2(1+|a_V|^2)\right]|M_F|^2 + \left[|C_T|^2(1+|a_T|^2) + |C_A|^2(1+ \right.$$
$$\left. + |a_A|^2)\right]|M_{GT}|^2 + |C_P|^2(1+|a_P|^2)|M_P|^2 + 2\,\mathrm{Re}\left[C_P C_S^*(a_P + a_S^*)M_P M_F^*\right]$$

$$B = \left[|C_S|^2(1+|a_S|^2) - |C_V|^2(1+|a_V|^2)\right]|M_F|^2 - \tfrac{1}{3}\left[|C_T|^2(1+|a_T|^2) - |C_A|^2(1+ \right.$$
$$\left. + |a_A|^2)\right]|M_{GT}|^2 + |C_P|^2(1+|a_P|^2)|M_P|^2 + 2\,\mathrm{Re}\left[C_P C_S^*(a_P + a_S^*)M_P M_F^*\right]$$

$$C = 2\,\mathrm{Re}\left[C_S C_V^*(1+a_S a_V^*)|M_F|^2 + C_T C_A^*(1+a_T a_A^*)|M_{GT}|^2 + C_P C_V^*(a_P + a_V^*)M_P M_F^*\right]$$

$$D = 2\,\mathrm{Im}\big[C_S C_V^*(1+a_S a_V^*)|M_F|^2 - \tfrac{1}{3} C_T C_A^*(1+a_T a_A^*)|M_{GT}|^2 + C_P C_V^*(a_P + a_V^*)M_P M_F^*\big]$$

$$I = 2\,\mathrm{Re}\big[(|C_S|^2 a_S - |C_V|^2 a_V)|M_F|^2 + (|C_T|^2 a_T - |C_A|^2 a_A)|M_{GT}|^2 + |C_P|^2 a_P |M_P|^2 + C_P C_S^*(1+a_P a_S^*)M_P M_F^*\big]$$

$$J = 2\,\mathrm{Re}\big[(|C_S|^2 a_S + |C_V|^2 a_V)|M_F|^2 - \tfrac{1}{3}(|C_T|^2 a_T + |C_A|^2 a_A)|M_{GT}|^2 + |C_P|^2 a_P |M_P|^2 + C_P C_S^*(1+a_P a_S^*)M_P M_F^*\big]$$

$$K = 2\,\mathrm{Im}\big[C_S C_V^*(a_S + a_V^*)|M_F|^2 + C_T C_A^*(a_T + a_A^*)|M_{GT}|^2 + C_P C_V^*(1+a_P a_V^*)M_P M_F^*\big]$$

$$L = 2\,\mathrm{Re}\big[C_S C_V^* a_V^* + C_V C_S^* a_S^*)|M_F|^2 - \tfrac{1}{3}(C_T C_A^* a_A^* + C_A C_T^* a_T^*)|M_{GT}|^2 + C_P C_V^*(1+a_P a_V^*)M_P M_F^*\big].$$

Abstract

Tests for time reversal invariance and for the two-component theory of Feynman and Gell-Mann are discussed on the basis of allowed beta transitions. Experiments are suggested, in which the electron-neutrino angular correlation and electron polarization are to be measured simultaneously. Coulomb effects have been taken into account.

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Boundary Conditions at Infinity for Physical Theories

by

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1. The Cauchy problem is the most natural for hyperbolic partial differential equations. When dealing with physical problems, we are, however, often interested in solutions of field equations with given sources when nothing is known about initial conditions. A whole set of fields corresponds, in general, to given sources and, in order to arrive at a unique solution of the problem, we must specify some additional condition. For linear field equations this condition may consist in prescribing the form of Green's function (e. g. retarded, advanced, etc.). If we investigate the field in the whole (unbounded) space-time we can ensure uniqueness by specifying some appropriate *boundary conditions* at spatial infinity. The latter approach has the advantage of being applicable to non-linear theories, such as the theory of general relativity. These boundary conditions, first formulated for a periodic scalar field by Sommerfeld [1], have a definite physical meaning. E. g., the "Ausstrahlungsbedingung" of Sommerfeld means that the system can lose its energy in the form of radiation and that no waves are falling on the system from the exterior.

The purpose of this paper is to formulate boundary conditions for scalar and Maxwell theories in a form which exhibits their physical meaning and is proper to a generalization for the gravitational case.

2. Let us first take the *scalar wave equation*

$$\Delta\varphi - \frac{\partial^2\varphi}{\partial t^2} = -4\pi\varrho\,, \tag{1}$$

and assume $\varrho(\vec{r}, t)$ to be a regular function vanishing outside a bounded 3-dimensional region V. The retarded solution of (2) can be written in the form

$$\varphi(\vec{r}, t) = \int_V \frac{\varrho(\vec{r}', t-R)}{R}\, dV', \qquad R = |\vec{r} - \vec{r}'|\,. \tag{2}$$

From (2) we obtain the following asymptotic values of φ and its derivatives:

$$\varphi = r^{-1}\int_V \varrho(\vec{r}', t-R)\,dV' + O(r^{-2}),$$

$$\varphi_{,\sigma} = k_\sigma r^{-1}\int_V \varrho_{,\sigma}(\vec{r}', t-R)\,dV' + O(r^{-2})\ *),\tag{3}$$

where

$$k^\sigma = (1, n^s), \qquad n^s = x^s/r \tag{4}$$

is a null vector field.

Now, we can formulate the following boundary conditions to be imposed on solutions of (1):

$$\varphi = O(r^{-1})\,; \tag{5}$$

(6) *there exists a function* $\psi = O(r^{-1})$ *such that* $\varphi_{,\nu} = \psi k_\nu + O(r^{-2})$, *where* k_ν *is given by* (4).

We see from Eqs. (3) that every retarded solution of (1) fulfills (5) and (6). Conversely, if condition (6) is fulfilled, then φ satisfies Sommerfeld's radiation condition

$$\lim_{r\to\infty} rk^\nu\varphi_{,\nu} = \lim_{r\to\infty} r(\partial\varphi/\partial t + \partial\varphi/\partial r) = 0\,.$$

Thus, the wave equation with a spatially bounded source has always one, and only one, solution fulfilling our conditions (5) and (6).

If we replace (4) by $k^\sigma = (1, -n^s)$ we obtain the conditions which characterize advanced solutions of (1).

Let us introduce the energy-momentum density tensor of the field φ:

$$T^\nu_\mu = L\delta^\nu_\mu - \varphi_{,\mu}\partial L/\partial\varphi_{,\nu}, \qquad \text{where} \qquad L = -\eta^{\varrho\sigma}\varphi_{,\varrho}\varphi_{,\sigma}/8\pi\,.$$

Taking into account the asymptotic expressions of $\varphi_{,\nu}$ we have $L = O(r^{-3})$ and

$$4\pi T_{\mu\nu} = \psi^2 k_\mu k_\nu + O(r^{-3})\,. \tag{7}$$

The asymptotic form of $T_{\mu\nu}$ resembles the energy-momentum tensor of a perfect fluid with vanishing rest mass. Further, we can obtain from (7) the time rate of radiated energy and momentum:

$$4\pi W_\mu = 4\pi\oint_S T^s_\mu n^s dS = \oint_S \psi^2 k_\mu dS\,.$$

*) $\Phi_A = O(r^{-k})$ means that there exists a constant M such that, for a sufficiently large r, we have $|\Phi_A| < Mr^{-k}$; Greek indices run from 0 to 3, Latin ones — from 1 to 3; $x^0 = t$, $(x^1, x^2, x^3) = \vec{r}$; a comma followed by an index denotes differentiation. Summation convention will be used throughout. Indices will be raised by means of the Galilean metric tensor $\eta^{\alpha\beta}$ ($\eta^{00} = 1$, $\eta^{i0} = 0$, $\eta^{ik} = -\delta^{ik}$). Square brackets stand for alternation, e. g., $\Phi_{[\mu\nu\varrho]} = \Phi_{\mu\nu\varrho} + \Phi_{\nu\varrho\mu} + \Phi_{\varrho\mu\nu} - \Phi_{\varrho\nu\mu} - \Phi_{\nu\mu\varrho} - \Phi_{\mu\varrho\nu}$.

The integrals are to be taken over the surface of a sphere "at infinity". The condition (6) with Eq. (4) ensures that $W_0 \geqslant 0$.

3. The situation is somewhat more complicated in *electrodynamics* because of the gauge-invariance. Maxwell's equations

$$f^{\mu\nu}{}_{,\nu} = -4\pi j^{\mu}, \qquad f_{\mu\nu} = A_{[\nu,\mu]} \tag{8}$$

can be reduced to four wave equations

$$\Delta A^{\mu} - \frac{\partial^2 A^{\mu}}{\partial t^2} = -4\pi j^{\mu} \tag{9}$$

if one imposes on potentials the Lorentz condition

$$A^{\nu}{}_{,\nu} = 0\,. \tag{10}$$

To A^{ν}, satisfying Eqs. (9) and (10), we can apply conditions (5), (6). It would perhaps be more satisfactory if we formulated the boundary conditions in a way involving only the *field* $f_{\mu\nu}$. However, our actual conditions will be more suited for a straightforward generalization to Einstein's theory. The current j^{μ} now satisfies the same regularity and boundness conditions as ϱ in the former case.

We formulate the boundary conditions as follows: *there exists a potential* A^{μ} *satisfying*

$$A^{\mu} = O(r^{-1}) \tag{11}$$

(12) *and four functions* $B_{\mu} = O(r^{-1})$ *such that* $A_{\varrho,\sigma} = B_{\varrho} k_{\sigma} + O(r^{-2})$, *and*

$$B_{\varrho} k^{\varrho} = O(r^{-2})\,. \tag{13}$$

It must be noted that there are many functions A_{α} which satisfy Maxwell's Eqs. (8) and conditions (11)-(13), but all these potentials represent the same electromagnetic field.

From Eqs. (12), (13) we obtain the asymptotic form of the field:

$$f_{\mu\nu} \cong k_{\mu} B_{\nu} - k_{\nu} B_{\mu}\,, \qquad k_{\varrho} B^{\varrho} \cong 0 \;^{*}) \tag{14}$$

or, in vector notation,

$$\vec{E} \cong (\vec{B} \times \vec{n}) \times \vec{n}\,, \qquad \vec{H} \backsimeq \vec{B} \times \vec{n}\,, \qquad B^{\nu} = (B^0, \vec{B}), \qquad k^{\nu} = (1, \vec{n})\,.$$

Eqs. (14) represent a system of "gauge-invariant" boundary conditions. The electromagnetic field has asymptotically the form of a plane wave. For the energy-momentum density tensor

$$4\pi T_{\mu\nu} = \tfrac{1}{4}\,\eta_{\mu\nu} f_{\varrho\sigma} f^{\varrho\sigma} - f_{\mu\varrho} f_{\nu}{}^{\varrho}$$

we obtain the expression

$$4\pi T_{\mu\nu} = -B_{\varrho} B^{\varrho} k_{\mu} k_{\nu} + O(r^{-3})\,.$$

*) We shall sometimes write $\Phi \cong 0$ instead of $\Phi = O(r^{-2})$.

k_ν being a null vector, it follows from (13) that $B_\varrho B^\varrho \leqslant 0$ for sufficiently large r.

The total charge e contained in the field can be calculated by means of Gauss' law

$$4\pi e = \oint_S f^{k0} n^k dS .$$

Though f^{k0} contains terms going as $1/r$, nevertheless e is finite by virtue of (14).

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Radiation and Boundary Conditions in the Theory of Gravitation

by

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The aim of this paper is to discuss the connection between the problem of gravitational radiation and the boundary conditions at infinity. We shall deal with the concept of energy and momentum in Einstein's general relativity and propose a prescription for computing the total radiated energy. A connection between our radiation conditions and the definitions of gravitational radiation by Pirani and Lichnerowicz is shown in section 5.

1. In physics we are ordinarily interested in conservation laws which have an integral character. A classical conserved quantity is a functional $f[\sigma]$ depending on a space-like hypersurface σ. A conservation law is a statement that, by virtue of the equations of motion, f, in fact, does not depend on σ. As is known, in general relativity the energy-momentum tensor of matter $\mathfrak{T}_{\mu\nu}$ does not by itself lead to an integral conservation law. However, if we introduce an energy-momentum pseudotensor of the gravitational field $\mathfrak{t}_\mu^\nu = (\delta_\mu^\nu \mathfrak{G} + g^{\varrho\sigma}{}_{,\mu}\partial\mathfrak{G}/\partial g^{\varrho\sigma}{}_{,\nu})/2\varkappa$, then the sum $\mathfrak{T}_\mu^\nu + \mathfrak{t}_\mu^\nu$ is divergenceless by virtue of Einstein's equations *). Einstein's tensor density $\mathfrak{G}_\mu^\nu = \sqrt{-g}(R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R)$ can namely be written in the form

$$\mathfrak{G}_\mu^{\ \nu} \equiv \varkappa(\mathfrak{t}_\mu^{\ \nu} + \mathfrak{U}_\mu^{\ \lambda\nu}{}_{,\lambda}), \tag{1}$$

where the "superpotentials" $\mathfrak{U}_\mu^{\ \nu\lambda}$ are given in [1]

$$2\varkappa\mathfrak{U}_\mu^{\ \nu\lambda} \equiv \sqrt{-g}\, g^{\sigma[\varrho}\delta_\mu^{\nu}g^{\lambda]\tau}g_{\varrho\sigma,\tau} \equiv -2\varkappa\mathfrak{U}_\mu^{\ \lambda\nu}. \tag{2}$$

*) We shall use the notations of the preceding paper; $g_{\mu\nu}$ will denote the metric tensor of the Riemannian space-time V_4. Gothic letters denote tensor densities and also "pseudoquantities" such as the superpotentials.

If the Einstein equations

$$G_{\mu\nu} = -\varkappa T_{\mu\nu} \tag{3}$$

are satisfied, then Eqs. (1) and (2) imply

$$\mathfrak{T}_\mu^\nu + \mathfrak{t}_\mu^\nu = \mathfrak{U}_\mu^{\nu\lambda}{}_{,\lambda}, \quad \text{thus} \quad (\mathfrak{T}_\mu^\nu + \mathfrak{t}_\mu^\nu)_{,\nu} = 0\,. \tag{4}$$

The functions $\mathfrak{t}_\mu^\nu$ are not components of a tensor density (equivalence principle) and many physicists (e. g., Schrödinger [2]) have raised doubts as to their physical meaning. Einstein [3] and F. Klein [4] formulated some conditions which enable us to consider the integrals

$$P_\mu[\sigma] = \int_\sigma (\mathfrak{T}_\mu^\nu + \mathfrak{t}_\mu^\nu)\, dS_\nu = \oint_S \mathfrak{U}_\mu^{\nu\lambda} dS_{\nu\lambda} \tag{5}$$

as representing the total energy and momentum of the system: matter and gravitational field. These conditions can be summarized as follows. Let us take an isolated system of masses ($T_{\mu\nu} = 0$ outside a bounded 3-region) and assume the existence of co-ordinates such that [5]

$$g_{\mu\nu} = \eta_{\mu\nu} + O(r^{-1}), \quad g_{\mu\nu,\varrho} = O(r^{-2})\,, \tag{6}$$

where r denotes the distance measured along geodesics from a fixed point on a space-like σ. Eqs. (6) have a double meaning: they constitute a system of boundary conditions, and they distinguish a set of co-ordinate systems ("Galilean at infinity").

Using (4) it can be easily proved that: $1^0 - P_\mu[\sigma]$, calculated from (5) in a co-ordinate system satisfying (6), is always finite and does not depend on σ; $2^0 - P_\mu$ does not depend on co-ordinate changes which do not alter (6) and reduce to an identity for $r \to \infty$; $3^0 - P_\mu$ is a vector with respect to linear orthogonal transformations. The proof is based on the vanishing of the integral

$$p_\mu = \int_\Sigma (\mathfrak{T}_\mu^\nu + \mathfrak{t}_\mu^\nu)\, dS_\nu \tag{7}$$

taken over a *time-like* "cylindrical" hypersurface Σ at spatial infinity (note that S appearing in (5) is the intersection of Σ and σ). The vanishing of these integrals is ensured by (6) ($\mathfrak{t}_\mu^\nu$ is quadratic in $g_{\mu\nu,\varrho}$) and our assumption on $T_{\mu\nu}$. The integral (7) can eventually be identified with the total energy and momentum radiated through Σ, and Lichnerowicz's boundary conditions (6) automatically exclude the existence of any radiation.

2. Comparison with electrodynamics suggests that radiation fields in general relativity should be characterized by $g_{\mu\nu,\varrho} \sim 1/r$, rather than by $g_{\mu\nu,\varrho} \sim 1/r^2$. However, if the integrals (7) do not vanish, the proof of the Einstein-Klein theorem is no longer valid and doubts as to the

meaning of (5) arise anew. We propose to generalize the boundary conditions (6) in such a way as to include radiation fields. We expect that these conditions will ensure the finiteness of P_μ and that P_μ will not change with co-ordinate transformations which reduce to an identity for $r \to \infty$ and preserve the *form* of the boundary conditions. The dependence of P_μ on σ will now correspond to the diminishing of total energy due to radiation.

Fock [6] proposes to normalize the co-ordinate systems by means of de Donder's relation

$$g^{\mu\nu}{}_{,\nu} = 0 \tag{8}$$

and imposes on $g_{\mu\nu}$ the radiation condition of Sommerfeld. We find this formulation somewhat stringent. In particular, we see no reason for restricting ourselves to harmonic co-ordinates only. There is no convincing argument for writing the Schwarzschild line element in harmonic co-ordinates instead of, say, in isotropic ones.

We generalize the conditions of Fock along the lines presented in the preceding paper. First, introduce a null vector field k_ν defined as follows. Let n^ν be a unit space-like vector lying in σ, perpendicular to the "sphere" $r = \text{const.}$, and pointing outside it. We put $k^\nu = n^\nu + t^\nu$, where t denotes a unit time-like vector normal to σ, such that $t^0 > 0$.

Now, we formulate the following boundary conditions to be imposed on gravitational fields due to isolated systems of matter: *there exist co-ordinate systems and functions* $h_{\mu\nu} = O(r^{-1})$ *such that*

$$g_{\mu\nu} = \eta_{\mu\nu} + O(r^{-1}), \qquad g_{\mu\nu,\varrho} = h_{\mu\nu} k_\varrho + O(r^{-2}), \tag{9}$$

$$(h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\varrho\sigma}h_{\varrho\sigma})k^\nu = O(r^{-2}). \tag{10}$$

These conditions correspond to Sommerfeld's "Ausstrahlungsbedingung"; we obtain the "Einstrahlungsbedingung" assuming n^ν to be a normal pointing inward the sphere $r = \text{const}$. Relations (9), (10) are *weaker* than (6); this means that every field fulfilling (6) satisfies also our conditions (9), (10). The class of co-ordinate systems distinguished by (9) and (10) is larger than that defined by (6). Eq. (10) restricts the co-ordinate systems to those which are asymptotically harmonic; however, it must be noted that isotropic co-ordinates used in the Schwarzschild V_4 are asymptotically harmonic in our meaning.

Strictly speaking the correctness of conditions (9) and (10) might be inferred only if it were possible to show that Einstein's equations with bounded sources have always exactly one solution satisfying (9), (10). But it is not an easy task to prove this theorem.

3. We shall now present some consequences of (9) and (10). First of all, we must examine the convergence of energy integrals (5). The superpotentials are linear in $g_{\mu\nu,\varrho}$ and thus go as $1/r$; we must therefore

show that the terms behaving as $1/r$ cancel out in the surface integral (5). Indeed, the surface element dS_λ is proportional to $n_{[\lambda}t_{\nu]} = n_{[\lambda}k_{\nu]}$, and the terms in question in (5) can be written as $\eta^{\sigma[\varrho}\delta^{\nu}_{\mu}\eta^{\lambda]\tau}h_{\varrho\sigma}k_{\tau}k_{[\nu}n_{\lambda]}$. Taking into account (10), we verify that this expression does vanish.

Let us take a co-ordinate transformation

$$x^\nu \to x'^\nu = x^\nu + a^\nu(x) \tag{11}$$

fulfilling

$$a^\nu = o(r), \qquad a_{\nu,\mu} = b_\nu k_\mu + O(r^{-2}) \tag{12}$$

where

$$a_\nu = \eta_{\nu\mu}a^\mu, \qquad b_\nu = O(r^{-1}),$$

and

$$a_{\nu,\mu\varrho} = b_{\nu,\mu}k_\varrho + O(r^{-2}), \qquad b_{\nu,\varrho} = O(r^{-1}). \tag{13}$$

From (13) follows the existence of functions $c_\nu = O(r^{-1})$ such that

$$b_{\nu,\mu} = c_\nu k_\mu + O(r^{-2}). \tag{14}$$

Co-ordinate transformations (11) satisfying (12) and (13) preserve the form of our boundary conditions; this can be easily seen from the transformation formulae for $g_{\mu\nu}$ and $h_{\mu\nu}$:

$$\begin{aligned} g'_{\mu\nu}(x') &\cong g_{\mu\nu}(x) + b_\mu k_\nu + b_\nu k_\mu, \\ h'_{\mu\nu}(x') &\cong h_{\mu\nu}(x) + c_\mu k_\nu + c_\nu k_\mu. \end{aligned} \tag{15}$$

Computing the superpotentials in both co-ordinate systems and taking into account the relations (9)-(15) we obtain

$$\mathfrak{U}'^{\nu\lambda}_{\mu}k'_\nu n'_\lambda = \mathfrak{U}^{\nu\lambda}_{\mu}k_\nu n_\lambda + O(r^{-3}).$$

Therefore, the total energy and momentum P_μ are well defined by (5) and the boundary conditions (9), (10). It must be noted that our prescription demands that the *calculation* of P_μ should be performed by means of (5) using co-ordinates which satisfy Eqs. (9) and (10). This does not at all mean that the energy is only a property of the co-ordinate system. The vector $P_\mu[\sigma]$ constitutes a *global* characteristic of the field and it is only for computational purposes that we must appeal to (9), (10).

4. The total energy and momentum p_μ radiated between two hypersurfaces σ and σ' is given by (7), or by

$$p_\mu = P_\mu[\sigma] - P^{\mathrm{I}}_{\mu}[\sigma'] = \int_{\Sigma} \mathfrak{t}^{\nu}_{\mu} dS_\nu$$

($T_{\mu\nu}$ vanishes on Σ). The boundary conditions enable the estimation of p_μ; we have, indeed,

$$\mathfrak{t}^{\nu}_{\mu} = \tau k_\mu k^\nu + O(r^{-3}), \tag{16}$$

where

$$4\varkappa\tau = h^{\mu\nu}(h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\varrho\sigma}h_{\varrho\sigma})\,. \tag{17}$$

τ is invariant with respect to transformation (15) and is *non-negative* by virtue of (10); therefore $p_0 \geqslant 0$. The existence of radiation is characterized by $p_\mu \neq 0$.

We could also take a more general case, including the electromagnetic field. The boundary conditions for $g_{\mu\nu}$ should be supplemented by those for $f_{\varrho\sigma}$ given in the preceding paper.

We obtain in this case $\mathfrak{T}_\mu^{\ \nu} + \mathfrak{t}_\mu^{\ \nu} = \bar{\tau}k_\mu k^\nu + O(r^{-3})$, $0 \leqslant \bar{\tau} = O(r^{-2})$.

5. Pirani [7] and Lichnerowicz [8] recently proposed definitions of pure radiation fields. It may be interesting to compare their definitions with our approach. Let us admit the additional but reasonable assumption that the second derivatives of $g_{\mu\nu}$ also go to 0 as $1/r$ and that $g_{\mu\nu,\varrho\sigma} \cong h_{\mu\nu,\sigma}k_\varrho$. From $h_{\mu\nu,\sigma}k_\varrho \cong h_{\mu\nu,\varrho}k_\sigma$ there follows the existence of functions $i_{\mu\nu} = O(r^{-1})$ such that

$$g_{\mu\nu,\varrho\sigma} \cong i_{\mu\nu}k_\varrho k_\sigma\,, \qquad (i_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\varrho\sigma}i_{\varrho\sigma})k^\nu \cong 0\,. \tag{18}$$

For the curvature tensor we get

$$R \cong \tfrac{1}{2}k_{[\mu}i_{\nu][\varrho}k_{\sigma]}\,. \tag{19}$$

The principal part of $R_{\mu\nu\varrho\sigma}$ has therefore the same form as a discontinuity of the Riemann tensor [9] and is thus of type II in the Petrov-Pirani classification [7].

The terms proportional to $1/r$ in $R_{\mu\nu}$ cancel out by virtue of (10). Conversely, $R_{\mu\nu} \simeq 0$ and Eq. (18) imply $R_{\mu\nu\varrho\sigma} \simeq 0$ unless $k_\nu k^\nu = 0$. If we take into account the electromagnetic field, Einstein's equations can be written in the form

$$R_{\mu\nu} = \varrho k_\mu k_\nu + O(r^{-3})\,, \qquad \varrho = O(r^{-2})\,. \tag{20}$$

Moreover, it follows from (19) that

$$k_{[\mu}R_{\nu\varrho]\sigma\tau} \cong 0\,, \qquad k^\mu R_{\mu\nu\varrho\sigma} \cong 0\,. \tag{21}$$

If one replaces the asymptotic equalities $\simeq$ by strict ones, then Eqs. (20) and (21) become Lichnerowicz's conditions [8] characterizing a pure radiation field. The definitions of Lichnerowicz and Pirani concern the idealized case of pure radiation. Actual metrics approach these radiation fields only in the limit $r \to \infty$ (wave zone).

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On the Isobars of the Nucleon in the Fixed-Source Theory

by

W. KRÓLIKOWSKI

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The aim of this paper is to add some remarks to the discussion of isobars of the nucleon in the fixed-source pion theory [1].

In this theory the Hamiltonian has the form

$$H = M + H^M + H^{\mathrm{Int}}, \tag{1}$$

where

$$H^M = \tfrac{1}{2} \int d_3x [\pi_j^2 + c^2 \varphi_j (-\Delta + \mu^2) \varphi_j], \tag{2}$$

$$H^{\mathrm{Int}} = g\sigma_i\tau_j \int d_3x \varrho \partial_i \varphi_j = -g\sigma_i\tau_j \int d_3x \partial_i \varrho \varphi_j \tag{3}$$

and M is the bare mass of the nucleon.

Using the orthonormal expansions

$$\varphi_j(\vec{x}) = \sum_{\lambda,l,m} q^{(j)}_{\lambda lm} u_{\lambda l}(r) Y_{lm}(\vartheta, \varphi), \tag{4}$$

$$\pi_j(\vec{x}) = \sum_{\lambda,l,m} p^{(j)}_{\lambda lm} u_{\lambda l}(r) Y^*_{lm}(\vartheta, \varphi), \tag{5}$$

we separate all degrees of freedom with $l \neq 1$ [2] (we assume $\varrho(\vec{x}) = \varrho(r)$). If we introduce the notation

$$q_{\lambda ij} = \begin{cases} \dfrac{1}{\sqrt{2}} (q^{(j)}_{\lambda 11} + q^{(j)*}_{\lambda 11}) & i = 1 \\ \dfrac{1}{i\sqrt{2}} (q^{(j)}_{\lambda 11} - q^{(j)*}_{\lambda 11}) & i = 2 \\ q^{(j)}_{\lambda 10} & i = 3 \end{cases} \tag{6}$$

and

$$p_{\lambda ij} = \begin{cases} \dfrac{1}{\sqrt{2}} (p^{(j)}_{\lambda 11} + p^{(j)*}_{\lambda 11}) & i = 1 \\ \dfrac{i}{\sqrt{2}} (p^{(j)}_{\lambda 11} - p^{(j)*}_{\lambda 11}) & i = 2 \\ p^{(j)}_{\lambda 10} & i = 3, \end{cases} \tag{7}$$

we obtain the following formulae for the remainder of the Hamiltonian (1) corresponding to $l = 1$

$$H_{l=1} = M + \sum_{\lambda} \tfrac{1}{2}(p^2_{\lambda ij} + \omega^2_{\lambda\lambda} q^2_{\lambda ij}) - \sigma_i \tau_j \sum_{\lambda} \bar{g}_\lambda q_{\lambda ij} + \sum_{\lambda \neq \lambda'} \tfrac{1}{2} \omega_{\lambda\lambda'} q_{\lambda ij} q_{\lambda' ij}, \tag{8}$$

where

$$\omega^2_{\lambda\lambda'} = \int_0^\infty r\,dr\, u_{\lambda 1} \hat{\omega}^2_{l=1} u_{\lambda' 1}, \tag{9}$$

$$\bar{g}_\lambda = g \sqrt{\frac{4\pi}{3}} \int_0^\infty r^2 dr \frac{d\varrho}{dr} u_{\lambda 1} \tag{10}$$

and

$$\hat{\omega}^2_{l=1} = c^2 \left(-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{2}{r^2} + \mu^2 \right). \tag{11}$$

Now, we assume that the complete orthonormal system of radial functions $u_{\lambda 1}(r)$ contains the actual radial function of the pion cloud surrounding the bare nucleon. Denoting this function by $u_{01}(r)$ and abbreviating

$$q_{ij} = q_{0ij}, \quad p_{ij} = p_{0ij}, \quad \omega^2 = \omega^2_{00}, \quad \bar{g} = \bar{g}_0, \tag{12}$$

we get

$$H_{l=1} = \overset{0}{H} + \overset{1}{H}, \tag{13}$$

where

$$\overset{0}{H} = M + \tfrac{1}{2}(p^2_{ij} + \omega^2 q^2_{ij}) - \bar{g}\sigma_i \tau_j q_{ij} + \sum_{\lambda \neq 0} \tfrac{1}{2}(p^2_{\lambda ij} + \omega^2_{\lambda\lambda} q^2_{\lambda ij}), \tag{14}$$

$$\overset{1}{H} = \sum_{\lambda} (\omega^2_{0\lambda} q_{ij} - \bar{g}_\lambda \sigma_i \tau_j) q_{\lambda ij} + \sum_{\substack{\lambda \neq \lambda' \\ (\lambda \neq 0 \neq \lambda')}} \tfrac{1}{2} \omega^2_{\lambda\lambda'} q_{\lambda ij} q_{\lambda' ij}. \tag{15}$$

We shall identify the operator

$$H^I = M + \tfrac{1}{2}(p^2_{ij} + \omega^2 q^2_{ij}) - \bar{g}\sigma_i \tau_j q_{ij} \tag{16}$$

(cf. [3]) with the Hamiltonian of the physical nucleon (bare nucleon + + pion cloud). The physical nucleon has, in this theory, the internal degrees of freedom described by operators q_{ij}, $p_{ij} = -i\hbar\partial/\partial q_{ij}$, σ_i, τ_j ($i = 1, 2, 3$, $j = 1, 2, 3$). The equation

$$(E - H^I)\Phi = 0 \tag{17}$$

determines the mass levels E and eigenstates Φ of the physical nucleon. These mass levels and eigenstates may be numerated by means of $\vec{J}^{I2}$, J^I_3, $\vec{T}^{I2}$, T^I_3, where

$$\vec{J}^I = \frac{\hbar}{2}\vec{\sigma} + \sum_j (q_{1j}, q_{2j}, q_{3j}) \times (p_{1j}, p_{2j}, p_{3j}) \tag{18}$$

and

$$\vec{T}^I = \frac{\hbar}{2}\vec{\tau} + \sum_i (q_{i1}, q_{i2}, q_{i3}) \times (p_{i1}, p_{i2}, p_{i3}) \tag{19}$$

are the total spin and total isospin of the physical nucleon. The excited eigenstates represent the isobars of the nucleon. The charge operator of the physical nucleon has the form

$$Q^I = \tfrac{1}{2}(T_3^I/\hbar + 1)\,. \tag{20}$$

The above results are independent of the strong coupling approach [1]. Many other methods may be tried *a priori* to solve Eq. (17).

If we use in particular the strong coupling approximation for our Eq. (17) we obtain the following mass levels

$$E_j = M - \frac{3}{2}\frac{\bar{g}^2}{\omega^2} + \frac{1}{4}\frac{(\hbar\omega)^2}{(\bar{g}/\omega)^2} j(j+1)\,, \tag{21}$$

where

$$J^I = T^I = j\hbar, \qquad j = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots\,, \tag{22}$$

and where $J_3^I \leqslant J^I$ and $T_3^I \leqslant T^I$ describe the degeneracy of mass levels.

If we assume that the radial function of the pion cloud u_{01} is connected with the source function $d\varrho/dr$ by means of the Eq.

$$\hat{\omega}_{l=1}^2 u_{01}(r) = \frac{1}{C^{1/2}}\frac{d\varrho(r)}{dr}\,, \tag{23}$$

where

$$C = \int_0^\infty r^2 dr \left(\frac{1}{\hat{\omega}_{l=1}^2}\frac{d\varrho}{dr}\right)^2 \tag{24}$$

is the normalization constant, we get

$$\omega^2 = \frac{1}{C}\int_0^\infty r^2 dr \frac{d\varrho}{dr}\frac{1}{\hat{\omega}_{l=1}^2}\frac{d\varrho}{dr} = \sqrt{\frac{3}{4\pi}}\frac{1}{C^{1/2}}\frac{\bar{g}}{g} \tag{25}$$

and

$$\frac{1}{4}\frac{(\hbar\omega)^2}{(\bar{g}/\omega)^2} = \frac{3}{4}\frac{\mu}{1/a}\frac{1}{f^2} m_\pi c^2\,, \tag{26}$$

where $\mu = m_\pi c/\hbar$, $f^2 = (\mu g/c)^2 4\pi\hbar c$ (normalized $f^2 = 0.08$) and

$$\frac{1}{4\pi a} = 4\pi c^4 C = 4\pi \int_0^\infty r^2 dr \left(\frac{c^2}{\hat{\omega}_{l=1}^2}\frac{d\varrho}{dr}\right)^2. \tag{27}$$

The constant a may be approximately treated as the radius of the core of the nucleon. From (21) and (26) we obtain

$$\Delta E_j = E_j - E_{j=1/2} = \frac{3}{4}\frac{\mu}{1/a}\frac{1}{f^2}\left[j(j+1) - \frac{3}{4}\right] m_\pi c^2\,. \tag{28}$$

This is exactly the result of Pauli and Dancoff [1].

The first isobar $j = \frac{3}{2}$ corresponds in this theory to the well-known resonance state $\frac{3}{2}, \frac{3}{2}$ formed in the scattering of pion and nucleon (Brueckner's isobar).

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On the Weak Interactions of Elementary Particles

by

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In a previous paper [1] a theory of strangeness of elementary particles was proposed. A consequence of this theory was the possibility of existence of the second neutral pion $\pi^{0\prime}$ being a charge singulet, whereas the "old" neutral pion π^0 belonged to the charge triplet $\pi^- \pi^0 \pi^+$ *). The aim of the present note is to put forward the role which this hypothetical pion may play in the problem of weak interactions *involving* pions. In the second part of this paper the consideration is extended to other weak interactions.

It is well known that the Gell-Mann rule

$$|\Delta S| = 1 \quad \text{or} \quad |\Delta T_3| = \tfrac{1}{2} \tag{1}$$

is satisfied for pion decays of hyperons, e. g. for

$$\Lambda^0 \to p + \pi^- \quad \text{and} \quad \Lambda^0 \to n + \pi^0 . \tag{2}$$

According to the theory described in [1], the electron and the neutrino form the charge doublet: $T = \frac{1}{2}$, $S = 0$ (corresponding to neutron-proton doublet), the μ^- meson is the charge singulet: $T = 0$, $S = 1$ (corresponding to Λ^0 singulet) and, therefore, the rule (1) is also valid for the decay of the charged pion

$$\pi^- \to \mu^- + \bar{\nu} , \tag{3}$$

whereas it is not satisfied in the unobserved process

$$\pi^- \to e^- + \bar{\nu} . \tag{4}$$

The above facts seem to suggest that the pion decay of hyperons and the decay (3) are caused by *direct* weak interactions between hype-

*) The existence of this second pion was recently suggested by A. Baldin (Birmingham preprint) in connection with strong pion interactions.

rons, nucleons and pions in the first case and between the μ mesons, neutrinos and charged pions in the second case, and that the Gell-Mann rule (1) is valid for these direct interactions *). Then the decay (3) is a direct process, whereas the decay (4) can only occur as an indirect one through the strong pion-nucleon and weak beta interactions.

The rule (1) being the "rule of change" may be replaced for these direct weak interactions by a rule of conservation, if we assume the existence of the second neutral pion $\pi^{0\prime}$. Namely, the rule (1) may be interpreted as an indication that the representation of pion field given by the charge triplet $\pi^-\pi^0\pi^+$ and the charge singulet $\pi^{0\prime}$ is not suitable for the weak interactions and that the proper representation is provided by two charge doublets

$$\Pi_a = \begin{cases} \frac{1}{\sqrt{2}}(\pi^0 + i\pi^{0\prime}), \\ \pi^+ \end{cases} \quad \Pi_a^* = \begin{cases} \frac{1}{\sqrt{2}}(\pi^0 - i\pi^{0\prime}) \\ \pi^- \end{cases} \qquad (a = 1, 2)\,. \tag{5}$$

Then, in these direct weak interactions the pions have

$$T = \tfrac{1}{2}, \qquad S = -1 \tag{6}$$

(Π_a $(a = 1, 2)$ have $S = -1$, $T_3 = \mp \frac{1}{2}$ and Π_a^* $(a = 1, 2)$ have $S = 1$, $T_3 = \pm\frac{1}{2}$) and the rule (1) is replaced by the rule

$$\Delta S = 0 \quad \text{or} \quad \Delta T_3 = 0\,. \tag{7}$$

Strong interactions can, of course, also occur in some weak processes (together with direct weak interactions), e. g. may be in decays

$$K^- \to \pi^0 + \pi^- \quad \text{and} \quad K^- \to \begin{cases} \pi^0 + \pi^- + \pi^0 \\ \pi^- + \pi^+ + \pi^- \end{cases} \tag{8}$$

and in similar ones for K^0 mesons. In the strong steps of weak processes the pions have $T = 1$, $S = 0$ and, therefore, the rule (7) must, in general, be violated for indirect weak processes.

Let us remark that the pions play a marked role in the above theory of weak interactions: other particles do not change their isospins and strangenesses in the transition from strong to weak interactions. It is connected with the fact that pions are the only bosons having (in strong interactions) the integral isospin ($\pi^{+*} = \pi^-$, while $\Sigma^{+*} \neq \Sigma^-$). But some pairs of fermions — antifermions, e. g. $(e^-, \bar{\nu})$ or $(n, \bar{p})$ might form boson-like states with integral isospin (charge singulets and triplets), which in transition to weak interactions would change their integral isospin into half-integral (charge doublets) in the same way as pions do. This case

*) The direct weak interactions between fermions and bosons were recently discussed and supported by H. Umezawa, M. Konuma and K. Nakagawa (Tokyo preprint) from the point of view of their space-time transformation properties.

seems to occur, e. g. for $(e^-, \bar{\nu})$ pair in the decay

$$\mu^- \to \nu + e^- + \bar{\nu} \tag{9}$$

and for $(n, \bar{p})$ pair in the capture

$$n + \mu^+ \to p + \bar{\nu}\,. \tag{10}$$

Now, we shall extend our consideration to other weak processes. Namely, we shall try to formulate some preliminary principles, which seem to govern the isospin properties of weak interactions.

1) Each direct weak interaction is essentially of Yukawa type in the sense that it is the product of two parts: the "source part" being a quadratic form of fermion or boson fields and the "boson-like part" built of boson fields or pairs of fermion fields.

2) In the boson-like part of the interaction the pions form the charge doublet Π_α; also the pairs (e^*_α, e_β) $\left(\text{where } e_\alpha = \begin{Bmatrix} e^- \\ \nu \end{Bmatrix}\right.$ and (N^*_α, N_β) $\left(\text{where } N_\alpha = \begin{Bmatrix} n \\ p \end{Bmatrix}\right.$ form charge doublets $(e^*e)_\alpha$ and $(N^*N)_\alpha$ built of $e^*_\alpha e_\beta$ and $N^*_\alpha N_\beta$ respectively, in a similar way as Π_α is built of $M_{\alpha\beta}$ (cf. (5) and [1]).

3) The interactions are invariant under rotations around the axis of charge in the isospin space. The rule of conservation (7) holds in all cases (if we take into account the above described changes of isospins).

The following direct weak interactions may be proposed as a tentative realization of the above preliminary principles. The notation of the paper [1] is adopted. The space-time transformation properties of the interactions (see footnote p. 418 and [2], [3]) are not discussed here. The problem of renormalization is also not taken into account.

a) The interaction

$$\bar{L}\xi_\alpha L C_{\alpha\beta} \tfrac{1}{2}(\tau_3 + 1)_{\beta\gamma}[g_1 \Pi_\gamma + g_2 (e^* e)_\gamma + g_3 (N^* N)_\gamma + g_4 K_\gamma] + \text{h. c.}, \tag{11}$$

where

$$\tfrac{1}{2}(\tau_3 + 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

gives	does not give
$\pi^- \to \mu^- + \bar{\nu}$	$\pi^0 \to \mu^- + e^+$
$\mu^- \to \nu + e^- + \bar{\nu}$	$\mu^- \to e^- + e^- + e^+$
$n + \mu^+ \to p + \bar{\nu}$	$n + \mu^+ \to n + e^+$
$K^- \to \mu^- + \bar{\nu}$	$\bar{K}^0 \to \mu^- + e^+$.

It is well known that the processes in the second column are not observed. The interaction (8) together with strong interactions leads to

the process

$$K^- \to (\Lambda^0 + \bar{p} \to \Lambda^0 + \bar{p} + \pi^0 \to K^- + \pi^0) \to \mu^- + \bar{\nu} + \pi^0 .$$

b) The interaction

$$\bar{B}\xi_a B C_{\alpha\beta}\tfrac{1}{2}(\tau_3 + 1)_{\beta\gamma} g \Pi_\gamma + \text{h. c.} \tag{12}$$

(acting in some cases, e. g. $\Lambda^\circ \to n + \pi^\circ$, together with strong interactions) gives the pion decays of hyperons and, together with strong interactions, leads to the processes

$$K^- \to (\Lambda^0 + \bar{p} \to p + \pi^- + \bar{p}) \to \begin{cases} \pi^0 + \pi^- \\ \pi^0 + \pi^0 + \pi^- \\ \pi^- + \pi^0 + \pi^- \end{cases}$$

and to analogical ones for K^0.

c) The interaction

$$\bar{L}\tau_i L (T_3)_{ij} [f_1 \bar{\Pi}\tau_j \Pi + f_2 \bar{\Pi}\tau_j (N^* N) + f_3 \bar{K}\tau_j (N^* N)] + \text{h. c.}, \tag{13}$$

where

$$T_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(and for meson field $\bar{M} = M^* \overleftrightarrow{\partial}_4$),

gives	does not give
$K^- \to \pi^0 + e^- + \nu^-$	$K^- \to \pi^- + e^- + e^+$
$\bar{K}^0 \to \pi^+ + e^- + \nu^-$	$\bar{K}^- \to \pi^0 + e^- + e^+ .$

The processes in the second column are not observed. The interaction (13), together with strong interactions, leads to the beta decay

$$n \to (p + \pi^-) \to p + e^- + \nu$$

and to the process

$$K^- \to (\Lambda^0 + \bar{p} \to \Lambda^0 + \bar{p} + \pi^0 \to K^- + \pi^0) \to e^- + \bar{\nu} .$$

The second process should be less probable than the process $K^- \to \pi^0 + e^- + \bar{\nu}$ and, therefore, difficult to discover.

Many other interactions satisfying our principles may, of course, be also written.

The author is indebted to Professor Abdus Salam for a valuable discussion.

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

Резюме статей

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В. ЗАВАДОВСКИЙ, АКСИОМАТИЧЕСКАЯ ХАРАКТЕРИЗАЦИЯ НЕКОТОРЫХ КОЛЕЦ ДЕЙСТВИТЕЛЬНЫХ ФУНКЦИЙ . . . . . стр. 355–360

Элемент a коммутативного кольца с реляцией $\geqslant 0$ называем грубым, если для всякого x $ax \geqslant 0$ влечет за собой $x \geqslant 0$.

Рассматриваются абстрактные кольца двух родов. Коммутативное кольцо с реляцией $\geqslant 0$ называем *действительным кольцом* типа α, если в нем существует по крайней мере, один грубый элемент и если

1α. $a \geqslant 0$ *и* $b \geqslant 0$ *влечет за собой* $ab \geqslant 0$ *и* $a+b \geqslant 0$,

2α. *если* $a \geqslant 0$ *и* $b \geqslant 0$, *а также если* a *или* b — *грубый элемент, то* $a+b$ *тоже грубый*

3α. $a^2 \geqslant 0$ *для всякого* a,

4α. *если,* b — *грубый элемент, а также если* $b+na \geqslant 0$ *для* $n = 1, 2, \ldots$, *то* $a \geqslant 0$.

5α. $a \geqslant 0$ *и* $-a \geqslant 0$ *влечет за собой* $a = 0$.

Коммутативное кольцо с единицей и с реляцией $\geqslant 0$ называем действительным кольцом типа β, если

1β. $a \geqslant 0$ *и* $b \geqslant 0$ *влечет за собой* $ab \geqslant 0$, $a+b \geqslant 0$, *а также* $a^{-1} \geqslant 0$, *если только* a^{-1} *существует,*

2β. *если* $a \geqslant 0$, *то элемент* $1+a$ *обратим,*

3β. $a^2 \geqslant 0$ *для всякого* a,

4β. *если* $1+na \geqslant 0$ *для* $n = 1, 2, \ldots$, *то* $a \geqslant 0$,

5β. $a \geqslant 0$ *и* $-a \geqslant 0$ *влечет за собой* $a = 0$.

Кольцо типа α расширяется однозначно к кольцу типа β. Доказаны две теоремы о представлении:

I. Всякое кольцо типа α можно представить изоморфно как некоторое кольцо действительных функций, в общем неограниченных, с областями в общем разнящимися от целого пространства. Изоморфизм сохраняет реляцию $\geqslant 0$, а ограниченные элементы переводятся в ограниченные.

Кольцо типа β называем *нормальным,* если к любым двум элементам $x, y \geqslant 0$ таким, что их сумма является обратимой, можно подобрать элементы $u, v \geqslant 0$ такие, что элементы $x+u$ и $y+v$ обратимы и $uv \geqslant 0$.

П. Всякое кольцо типа β нормальное изоморфно, при сохранении реляции $\geqslant 0$ с некоторым кольцом непрерывных функций, определенных на открытых плотных подмножествах некоторого бикомпактного топологического пространства Хаусдорффа.

К. СЕКЛЮЦКИЙ, О СТЯГИВАЕМОМ В СЕБЕ ПОЛИЭДРЕ КОТОРОГО НЕЛЬЗЯ МЕТРИЗОВАТЬ СИЛЬНО ВЫПУКЛЫМ СПОСОБОМ стр. 361—364

Метрику ϱ в пространстве X называем сильно выпуклой, если для всяких точек: $a, b \in X$ имеется точно один центр, т. е. такая точка c, что $\varrho(a, c) = \varrho(b, c) = \frac{1}{2}\varrho(a, b)$.

Всякое топологическое пространство, метризуемое сильно выпуклым способом, стягиваемо в себе к точке. Это условие, однако, не является достаточным, а примеры стягиваемых в себе пространств, которых нельзя метризовать сильно выпуклым способом, имеются даже среди абсолютных ретрактов.

К. Борсук выдвинул вопрос — существуют ли также полиэдры, обладающие такими свойствами. Автор настоящей работы доказывает, что полиэдр, сконструированный в одной из работ К. Борсука, будучи стягиваем в себе к точке, не метризуется сильно выпуклым способом.

С. МРУВКА, О СОЕДИНЕНИЯХ Q-ПРОСТРАНСТВ . . . . стр. 365—368

В работе приводится пример вполне регулярного ненормального пространства X, которое не является Q-пространством, но которое может быть представлено как сумма двух замкнутых подмножеств, являющихся Q-пространствами. Пространство X строится следующим образом:

Пусть N — пространство, состоящее из всех точек эвклидовой плотности, лежащих на прямой $y = 0$ и из всех точек, лежащих вне этой прямой, но с рациональными координатами. Окрестности точки p, лежащей на прямой $y = 0$ имеют $\{p\} \cup K$, где K — множество всех точек пространства N, лежащих в некотором открытом круге, касательном к прямой $y = 0$ в точке p; все остальные точки пространства N изолированы. Множество точек прямой $y = 0$ обозначим как R. Пространство N является небольшим изменением пространства Немыцкого; кроме того, в работе [2] доказано, что N является Q-пространством.

Пространство X получается из двух экземпляров пространства N посредством склеивания точек p множества R из первого экземпляра с точками $\pi(p)$ множества R из другого экземпляра, где π — некоторая перестановка множества R на себя.

В. ПОГОЖЕЛЬСКИЙ, СВОЙСТВА РЕШЕНИЙ ПАРАБОЛИЧЕСКОЙ СИСТЕМЫ УРАВНЕНИЙ С ЧАСТНЫМИ ПРОИЗВОДНЫМИ . . . стр. 369—373

В настоящем докладе автор представил некоторые свойства решений параболической системы уравнений в виде (1). Теорема 1 касается предельного свойства (7) решения (5), аналогичного интегралу Пуассона-Вейерштрасса.

Теоремы 2, 3 и 4 касаются производных порядка $m < M$ потенциала объемного заряда, представленного формулами (8). Эти производные удовлетворяют неравенствам (11), (12), (13).

Теоремы 5, 6, 7 и 8 касаются существования производных порядка M потенциала объемного заряда и их свойств, выраженных уравнениями (17), (18), (19), (21).

Теорема 9 дает решение задачи Коши для системы уравнений (26).

З. ЛУЩКИ, О k-НЕЗАВИСИМОМ ПОГРУЖЕНИИ ЭВКЛИДОВЫХ ПРОСТРАНСТВ . . . . . стр. 375–378

Пусть E_n обозначает n-мерное эвклидово пространство. Подмножество A пространства E_α (α натурально или равно ω) называем вслед за К. Борсуком ([1], 351) k-независимым, если каждое его подмножество, состоящее из не более, чем $k+2$ точек, линейно независимо. Пусть X будет сепарабельное, метрическое пространство, и пусть $D_k(X)$ обозначает наименьшее число $\alpha \leqslant \omega$ такое, что X гомеоморфно некоторому k-независимому подмножеству E_α.

В статье дано решение следующего вопроса К. Борсука: является ли число $D_k(E_n)$ конечным для каждой пары k, n натуральных чисел? Решение это положительно:

$$D_k(E_n) \leqslant \tfrac{1}{2}(k+1)[(k+2)(k-1)+2] \quad (k, n = 1, 2, \ldots).$$

Из замечания К. Борсука ([1], 355) и из решения этого вопроса следует, что если $\dim(X) \leqslant n$, то

$$D_k(X) \leqslant (k+1)[(k+2)n+1].$$

А. КИРКОР, ЗАМЕЧАНИЕ О ТОПОЛОГИЧЕСКОМ ПРОИЗВЕДЕНИИ НА ОТРЕЗОК ЧИСЛОВОЙ ПРЯМОЙ . . . . . стр. 379–381

Через A обозначим n-мерное эвклидово пространство ($n \geqslant 2$), с которого устранена одна точка. Пусть теперь B — любое открытое подмножество n-мерного эвклидова пространства, а I — открытый отрезок числовой прямой. Профессором Белецким был поставлен следующий вопрос: будут ли гомеоморфны A и B, если гомеоморфны топологические произведения $A \times I$ и $B \times I$. В работе доказывается, что решение этого вопроса отрицательно для $n \geqslant 3$. Для $n = 3$ построенный нами пример основан на результате Фокса и Артина [1]. Его конструкция легко обобщается на случай большего числа размерностей.

В. СЛОВИКОВСКИЙ, ПО ВОПРОСУ ТЕОРИИ ОПЕРАТОРНЫХ СИСТЕМ (II) . . . . . стр. 383–386

Предметом работы является обобщение результатов работы [1], а в особенности так наз. Основной Теоремы.

Вводится понятие так наз. *тар*-системы $\mathfrak{A}$, т. е. пары (S, X) состоящей из группы (или линейного пространства) X, а также из полугруппы гомоморфизмов S (линейных трансформаций), преобразующих некоторые подгруппы

(линейные подпространства) $G_A \subset X$; $A \in S$ в X таким образом, что:

$$(AB)x = A(Bx)$$

если только левая сторона существует.

Аналогично, как в [1] вводится понятие идеала $\mathfrak{T}$ и понятие фактор-*тар*-системы $\mathfrak{A}/\mathfrak{I}$. Основная теорема состоит из двух частей. Первая часть определяет зависимость между данной *тар*-системой $\mathfrak{A}$ и фактор-*тар*-системой $\mathfrak{A}/\mathfrak{I}$, где $\mathfrak{I}$ является идеалом A. Другая часть касается построения гомоморфизма фактор-*тар*-системы $\mathfrak{A}/\mathfrak{I}$, в фактор-*тар*-систему $\mathfrak{A}''/\mathfrak{I}''$ при помощи данного гомоморфизма $\mathfrak{A}'$ в $\mathfrak{A}''$.

Г. ВЖЕЦИОНКО, ОТРАЖЕНИЕ ВРЕМЕНИ И β-РАСПАД стр. 387—393

В работе рассматривается вопрос инвариантности теории слабых взаимодействий относительно отражения времени Вигнера. Исследование проводится на примере β-взаимодействия на основе общего оператора перехода для определенного процесса. Таким образом, главные выводы работы не связаны с какими либо приближениями типа теории возмущений. Исследуется структура вероятности перехода для β-процессов. Показано, что в общем, из условия инвариантности относительно отражения времени не вытекает исчезновение временных псевдоскаляров в вероятности перехода. Однако, если оператор перехода эрмитовый и векторы состояния — начальный и конечный — являются векторами состояния свободных частиц, тогда условие инвариантности относительно отражения времени не позволяет на выступание в вероятности перехода временных псевдоскаляров.

Б. КУХОВИЧ, ПРОВЕРКА ИНВАРИАНТНОСТИ ОТНОСИТЕЛЬНО ОТРАЖЕНИЯ ВРЕМЕНИ И ПРОВЕРКА ТЕОРИИ ФЕЙНМАНА И ГЕЛЛ-МАННА В РАЗРЕШЕННЫХ β-ПЕРЕХОДАХ . . . . . стр. 395—401

В работе рассматривается проверка инвариантности относительно отражения времени и проверка двухкомпонентной теории Фейнмана и Гелл-Манна на основании разрешенных β-переходов. Предлагаются испытания, в которых угловая корреляция электрон-нейтрино и поляризация электрона измеряются одновременно. Учтено влияние кулоновского поля ядра.

А. ТРАУТМАН, ГРАНИЧНЫЕ УСЛОВИЯ НА БЕСКОНЕЧНОСТИ ДЛЯ ФИЗИЧЕСКИХ ТЕОРИЙ ПОЛЯ . . . . . стр. 403—406

Рассматривается некоторая модификация условий излучения А. Зоммерфельда в теории скалярного поля и в электродинамике. Доказывается, что тензор энергии-импульса этих полей приближается в волновой зоне к тензору энергии идеальной жидкости с исчезающей массой покоя.

А. ТРАУТМАН, ИЗЛУЧЕНИЕ И ГРАНИЧНЫЕ УСЛОВИЯ В ТЕОРИИ ГРАВИТАЦИИ . . . . . стр. 407—412

Сформулированы граничные условия для гравитационного поля изолированной системы масс. Эти условия являются обобщением условий Фока

и Лихнеровича с одной стороны и модификацией условий Зоммерфельда — с другой. Доказывается, что эти условия дают возможность определить полную энергию и импульс системы и оценить величину излучаемой мощности.

Выяснено, что гравитационное поле в волновой зоне приближается к полю „чистого излучения", которое рассматривалось Пираним и Лихнеровичем.

В. КРУЛИКОВСКИЙ, О ИЗОБАРАХ НУКЛЕОНА В ТЕОРИИ О ЗАДАННОМ ИСТОЧНИКЕ . . . . . стр. 413—416

В работе даются некоторые замечания к дискуссии, касающейся изобар нуклеона в теории о заданном источнике. Существование этих изобар вытекает из теории без применения метода сильного сцепления. Эти изобары соответствуют промежуточным состояниям, образующимся во время рассеяния мезонов π на нуклеонах.

В. КРУЛИКОВСКИЙ, О СЛАБЫХ ВЗАИМОДЕЙСТВИЯХ ЭЛЕМЕНТАРНЫХ ЧАСТИЦ . . . . . стр. 417—421

В работе предлагается теория слабых взаимодействий. В этой теории особая роль предназначена мезонам π. Эти мезоны, после перехода от сильных взаимодействий к слабым меняют изотопический спин с целого на половинный. Получено пояснение для случая непроявления не наблюдаемых до сих пор слабых реакций.

TABLE DES MATIÈRES

Mathématique

1. W. Zawadowski, Axiomatic Characterization of some Rings of Real Functions 355
2. K. Sieklucki, On a Contractible Polytope which Cannot be Metrized in the Strong Convex Manner 361
3. S. Mrówka, On the Unions of Q-Spaces 365
4. W. Pogorzelski, Propriétés des solutions du système parabolique d'équations aux dérivées partielles 369
5. Z. Łuszczki, Sur le plongement k-indépendant d'espaces euclidiens 375
6. A. Kirkor, A Remark About Cartesian Division by a Segment 379
7. W. Słowikowski, On the Theory of Operator-Systems. II. . 383

Physique Théorique

8. J. Wrzecionko, Beta-Decays and Time Reversal 387
9. B. Kuchowicz, Tests for Time Reversal Invariance and Feynman-Gell-Mann Theory in Allowed Beta Transitions 395
10. A. Trautman, Boundary Conditions at Infinity for Physical Theories 403
11. A. Trautman, Radiation and Boundary Conditions in the Theory of Gravitation 407
12. W. Królikowski, On the Isobars of the Nucleon in the Fixed-Source Theory 413
13. W. Królikowski, On the Weak Interaction of Elementary Particles 417

Cena zł 20.—